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bosticj@bgsu.edu

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seanpyee@gmail.com

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mcgalliard@ucmo.edu

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Hope.marchionda@wku.edu

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ryan.fox@belmont.edu

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RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** is to continue to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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Exploring Teacher Learning

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(RE)CONSIDERING TEACHERS' PROMOTION OF THE STANDARDS FOR MATHEMATICAL PRACTICE

Jonathan D. Bostic
Bowling Green State University
bosticj@bgsu.edu

Gabriel Matney
Bowling Green State University
gmatney@bgsu.edu

Toni Sondergeld
Drexel University
tas365@drexel.edu

This study investigated mathematics teachers' teaching practices and the ways they promoted the Standards for Mathematical Practice (SMPs) before and after yearlong professional development (PD). Our research questions are: (1) To what degree did teachers' promotion of the SMPs change after yearlong PD focused in this area? (2) Were there any differences between cohorts and/or grade-bands in their promotion of the SMPs? Results express that teachers' promotion of the SMPs grew significantly during the PD and there were significant differences between elementary and secondary teachers.

Although professional development (PD) and the National Council of Teachers of Mathematics ([NCTM]; 2000) process standards have been researched for several decades, the adoption of the Common Core State Standards for Mathematics (CCSSM) and development of the Standards for Mathematical Practice (SMPs) provide a new and important context for study. It is not a foregone conclusion that the existence of mathematical practices such as the SMPs necessarily implies that teachers promote them during instruction (Bostic & Matney, 2014a; Hiebert et al., 2005). Moreover, it cannot be implied that teachers have made sense of them (Bostic & Matney, 2014b; Olson, Olson, & Capen, 2014). The present study provides evidence of the effects of yearlong PD on teachers' instruction, particularly in their promotion of the SMPs, which are central to doing and learning classroom mathematics (CCSSI, 2010; Koestler, Felton, Bieda, & Otten, 2013). It builds from past research (e.g., Bostic & Matney, 2014a) with inclusion of new data from two similar PD programs. Furthermore, the study provides research-based implications for mathematics educators who provide PD to teachers. Undergirding the present study are two sets of literatures: research on teachers implementing the SMPs and research on professional development.

Related Literature

Standards for Mathematical Practice

The SMPs are part of the Common Core State Standards for Mathematics (CCSSM) (Common Core State Standards Initiative [CCSSI], 2010). They offer characterizations of behaviors and habits that students should demonstrate while learning mathematics (Bostic &

Matney, 2016; CCSSI, 2010). The *Principles and Standards for School Mathematics* (NCTM, 2000) and *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) guided the descriptions of the SMPs. The literature is clear that teachers' instructional emphasis of the process standards, which promoted students' mathematical proficiency prior to the CCSSM, did not occur often (Hiebert et al., 2005). Initial research reports about CCSSM implementation suggests that K-12 teachers are struggling to make sense of the SMPs (Bostic & Matney, 2014b; Olson et al., 2014) much less weave the SMPs into their everyday instruction on the SMCs (Bostic & Matney, 2016). These findings suggest a need for research about professional development that enhances teachers' understanding of the SMPs and supports them to design and actualize instruction that makes the SMPs a part of their mathematics teaching.

Professional Development

We drew upon Guskey & Yoon's (2009) analysis of research about what works in PD when considering the design of the PD involved in this study. We sought to structure PD that adhered to the key features they found to be effective: (a) PD should have workshops focused on "research-based instructional practices" (p. 496) involving active-learning experiences for participants; (b) PD activities ought to encourage teachers to adapt a variety of practices to a specific content area, (c) PD should include a sufficient amount of time for teachers to make sense of the ideas and promote the application of these ideas during teachers' instruction; (d) PD ought to be structured and have sustained follow-up. A content-focused PD experience provided a space for teachers to apply a variety of practices to their classroom instruction. We utilized these features in tandem with the research-based work of NCTM's (2007) implementation standards for teaching and learning to provide teachers a conceptualization of teaching as sufficiently complex enough to promote student learning. The NCTM (2007) implementation standards define and emphasize the importance of worthwhile mathematical tasks, learning environment, and discourse. Past research has utilized these standards in PD. Boston (2012) detailed how focusing on implementing worthwhile tasks during a yearlong PD enhanced secondary teachers' knowledge, which in turn influenced their instructional practices. For example, after the yearlong PD they were able to identify elements of tasks with high cognitive demand and concurrently selected more tasks with high cognitive demand for their own instruction. Improving teachers' ability to select worthwhile tasks is not the only way to impact their instructional outcomes (Boston & Smith, 2009); supporting them to establish an effective learning environment and sustain mathematical discourse between students are also necessary to maximize students' opportunities to learn (NCTM, 2007). Our research builds upon Boston and others' work by adding a new layer into PD, the SMPs. The research questions for the present study are: (1) To what degree did teachers' promotion of the SMPs change after sustained (i.e., 100 or more hours) PD focused in this area? (2) Were there any differences between cohorts and/or grade-bands in their promotion of the SMPs?

Method

Context of the Professional Development

We aim to explore how teachers' instruction changed to promote the SMPs and connect this growth to PD projects. We focus on elementary and secondary teachers' experiences as influenced by four sets of teachers in sustained grant-funded professional development programs from one Midwest state. Three of those sets included cohorts of K-5 and grades 6-10 (i.e., Algebra 2) mathematics teachers who convened for a one-year program during 2012-2013, 2013-2014, and 2014-2015. These cohorts met for 100 hours during one calendar year. The fourth set included cohorts of K-5 and grades 6-8 mathematics teachers who convened for a two-year program (2014-2016) for a total of 256 PD hours (128 hours per year). For ease of reading, we name the set of K-5 and 6-8 cohorts from 2014-2016 "Apple" and "Blueberry" and the three sets of K-5 and 6-10 cohorts from 2012-2015 "Cherry". Generally speaking, the aim of the PD projects included (1) making sense of the SMPs, (2) exploring inquiry through NCTM's (2007) standards (i.e., worthwhile tasks, mathematical discourse, and appropriate learning environments), (3) implementing classroom-based tasks that aligned with the CCSSM, and (4) increasing mathematical knowledge and understanding. Teachers read and discussed chapters from NCTM books (e.g., *Mathematics Teaching Today* [NCTM, 2007]) and completed various assignments including journaling, writing, enacting, and reflecting on CCSSM-aligned mathematics lessons, and solving rich mathematics tasks. They also reflected on their mathematics instruction as well as the instruction of others implementing the CCSSM. Additionally, Apple and Blueberry cohorts engaged in lesson studies each semester, which were conducted at schools of participating teachers, while Cherry cohorts did not. Thus, the PD formats were fairly similar except for lesson study and number of hours met.

Participants

A total of 152 teachers participated in this study between the three PD programs. Table one shows teacher sample data by program they participated in and grade level they taught at the time of participation. Across the Apple cohort, 20 secondary teachers were part of the program. Thirty-four teachers composed the Blueberry cohort, (i.e., n= 23 elementary and n = 11 secondary). Ninety-four teachers participated in the Cherry cohorts; (n = 64 elementary and n= 35 secondary mathematics). Teachers for Apple, Blueberry, and Cherry cohorts came from urban, suburban, and rural school districts. All cohorts followed the same meeting format, used the same framework for the PD, but met in different parts of the Midwest state due to geographical constraints.

Table 1

Demographic Data for Teacher Participants

Demographic Variables	Frequency (%)
Program	
Apple	20 (13%)
Blueberry	34 (22%)
Cherry	98 (65%)
Grade Level	
Elementary	87 (57%)
Secondary	65 (43%)
Program x Grade Level	
Program A Elementary	0 (0%)
Program A Secondary	20 (100%)
Program B Elementary	23 (68%)
Program B Secondary	11 (32%)
Program C Elementary	64 (65%)
Program C Secondary	34 (35%)

Data Collection and Analysis

Teachers were asked to design, enact, and videotape one lesson when the PD began (i.e., pre-PD) and again near the end of the PD. For Cherry cohorts, this occurred after one year of PD

(100 hours), for Apple and Blueberry cohorts this occurred after two years of PD (256 hours). Depending on the grade level and the local school context of the teacher, the videos were as short as 25 minutes and as long as 65 minutes. Since our study focused on ways that teachers supported students' engagement in the SMPs during instruction, we investigated the videotapes as a means to report instructional changes made during the PD program. Such analysis approaches have been used in similar studies such as Boston (2012) and Boston and Smith (2009).

Data analysis required two parts. The first part involved watching the videotapes and reflecting on instruction using a protocol focused on the ways that teachers' instruction promoted the SMPs. Two mathematics education faculty as well as seven mathematics education graduate students watched the videotapes and conducted the analysis using a protocol validated for this purpose (see Bostic & Matney, 2016; Bostic, Matney, & Sondergeld, 2017). It provides look-fors that link mathematics teaching behaviors and the SMPs. For example, three aspects for the first SMP: Make sense of problems and persevere in solving them (CCSSI, 2010), include looking for the ways teachers (a) Involve students in rich problem-based tasks that encourage them to persevere in order to reach a solution, (b) Provide opportunities for students to solve problems that must have multiple solutions and/or strategies, and (c) Encourage students to represent their thinking while problem solving (Bostic & Matney, 2016). While there may be other aspects indicative of SMPs, the protocol provides an evidence-based framework for examining mathematics instruction using the SMP lens. Next, pairs of coders compared their observations with one another to gather interrater agreement. To maintain fidelity with use of the protocol, the team conducted meetings every few months with the sole purpose of establishing interrater agreement. The minimum threshold for interrater agreement is r_{wg} = .9 (James, Demaree, & Wolf, 1993). Interrater agreement exceeded the minimum threshold; it was as low as r_{wg} =.92 and as high as r_{wg} = 1.0]. Thus, we felt confident that our team was applying codes in a consistent manner.

The second part of data analysis focused on quantifying changes in the number and type of instructional opportunities related to the SMPs. The type and frequency of instructional opportunities related to each SMP were categorized. We then summed the pre-PD number of indicators for each SMP to create a grand total across all eight SMPs. The pre-PD grand total was compared to the post-PD grand total. Next, we completed a 2 Within, 2 X 3 Between

Factorial ANOVA to answer our research questions. The within independent variable was time (pre-PD and post-PD). Between independent variables were grade level taught (elementary or secondary) and PD program (Program Apple, Blueberry, or Cherry). The dependent variable was SMP score across all analyses.

Results

For RQ1, regardless of PD group or grade level taught, on average teachers expressed significantly more opportunities to promote the SMPs during post-PD instructional observations compared to pre-PD instructional observations; $F(1, 147) = 58.87, p<.001$. The effect size is large with partial $\eta^2 = .286$ indicating that 28.6% of the change in SMP scores is attributed to the PD. For RQ2, there were no significant differences by grade level taught (p=.465). However, there were significant differences by program; $F(2, 147) = 3.71, p=.027$. The effect size is small with partial $\eta^2 = .048$ indicating that only 4.8% of the variance in SMP scores can be attributed to program. This statistical difference between programs in average SMPs was noted at the post-PD observation time between only Apple (*M*=3.36, *SD*=2.11) and Cherry (*M*=5.37, *SD*=2.78) programs (p<.01). At pre-PD observation time, all programs performed statistically similar. On average, teachers increased 2.35 (*SD*=2.73) SMP indicators from pre-PD to post-PD; Cherry increased most (*M*=2.82, *SD*=2.78) followed by Blueberry (*M*=1.82, *SD*=2.45) and then Apple (*M*=0.95, *SD*=2.10).

Limitations

There are limitations to this study. Our sampling frame has limitations. Teachers from Apple cohorts volunteered to participate in the PD whereas the same is not true for Blueberry cohorts. Broadly speaking, more than half of the teachers from Blueberry cohorts were (a) required to attend by school or district-level personnel or (b) strongly encouraged by peers who decided to participate. Thus, those who are less motivated to complete long-term PD may have different outcomes making instructional changes. Moreover, many of the teachers from the Apple cohort participated in PD between 2012-2015 as part of the Cherry cohort. It is plausible that there may be a ceiling effect for average promotion of SMPs during instruction, which limited the mean growth for Apple teachers.

Importance of the Research

Taken collectively, these quantitative findings suggest that on average, teachers provided more opportunities for students to engage in the SMPs after the PD. All teachers showed growth in their promotion of SMPs after experiencing more than 100 hours of PD. There were no statistically significant differences across cohorts of teachers but there was a difference in the frequency with which elementary and secondary teachers promoted the SMPs during their instruction. These results have implications that connect research and practice. First, we noticed that instructional opportunities were clearly influenced by the implementation of teachers' choices of task, changes in learning environment, and ways discourse was promoted (see Bostic & Matney, 2014a; 2016 for discussion). Teachers internalized the standards for teaching and learning mathematics (NCTM, 2007) in ways that resonated with their instruction in the Common Core-era. Second, Guskey and Yoon's (2009) framework provided a means for us, as mathematics teacher educators, to frame our PD. This study adds convergent evidence that adhering to key features of PD leads to significant outcomes for PD participants.

In conclusion, the results broadly suggest that PD drawing upon Guskey and Yoon's key features as well as focusing on the CCSSM (CCSSI, 2010) and NCTM's Standards (2007) has potential to lead to changes in the way K-10 teachers designed and implemented mathematics instruction, as evidenced by teachers involved in this program. The SMPs do not dictate curriculum or teaching but they do provide ideas for engaging students in ways that promote mathematics proficiency during classroom instruction. PD may help mathematics teachers at all grade levels make sense of mathematics instruction that supports students' appropriate mathematical behaviors.

Endnote

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PRESERVICE TEACHERS EXPLORING PRIME FACTORIZATION

Ricela Feliciano-Semidei
The University of Montana
Ricela.feliciano-semidei@umontana.edu

Matt Roscoe
The University of Montana
roscoem@mso.umt.edu

Research on the understanding of prime factorization suggests that preservice teachers prefer computational methods to find factors. We present a teaching intervention implemented with K-8 preservice teachers in a public university in the Northwest US. Our intervention consisted of four worksheets that students discussed in small groups. The intervention intended to prompt students' recognition of existence and uniqueness of the prime factorization. Results from pre- and post- tests show that students combine different methodologies to find the factors of a number, but they begin to rely more heavily on the Fundamental Theorem of Arithmetic after experiencing the intervention.

The effort to fortify the mathematical understanding of preservice teachers is aimed at improving the teaching of mathematics at the elementary school level (Kajander, 2010; Brown, Thomas & Tolias, 2002; Manouchehri & Almohalwas, 2008). Research shows that many teachers in the United States struggle to understand the mathematical concepts that they need to teach (Ma, 1999). In particular, our investigation addresses teacher's struggle to understand elementary number theory, an important tool that aids in the teaching of factors and greatest common factor in fourth and sixth grade respectively.

Brown (2000) suggests that one of the principal elements for understanding elementary number theory is the deep understanding of prime factorization. However, the usefulness of prime factorization is under appreciated by preservice teachers, who tend not to fully understand the Fundamental Theorem of Arithmetic (FTA) (Zazkis & Campbell, 1996). This theorem states that every integer greater than one is either prime or it has a unique prime factorization. Research has revealed that participants' abilities to solve number theory problems related to the FTA can be improved after interventions (Feldman & Roscoe, 2015).

We explored the use of this theorem as a tool for finding the factors of a number – a skill that is taught in 4th grade according to the Common Core State Standards for Mathematics (CCSSM). Using the FTA to find factors can facilitate the recognition of the multiples of a number and the classification of a whole number as prime or composite, both mentioned as part of the 4th grade standard CCSS.MATH.CONTENT.4.OA.B.4. (CCSSI, 2010). A deep understanding of the FTA can help make connections for a better understanding of the greatest common factor and the least common multiple of a whole number by preservice elementary teachers (Feldman, 2012), a skill that is taught in 6th grade as part of the standard CCSS.MATH.CONTENT.6.NS.B.4. (CCSSI, 2010).

Theoretical Framework

The theories that helped shape the design of the teaching intervention include constructivist learning theories (Glasersfeld, 1989) and cooperative learning theories (Johnson & Johnson, 1999). Based on constructivism (Glasersfeld, 1989) students must actively investigate mathematics to form their own knowledge. The teaching intervention was implemented using cooperative learning groups (Johnson & Johnson, 1999), through which students worked collaboratively in small group settings and discussed their ideas. Participants were asked to actively explore, investigate and provide reasoning for their ideas (Glasersfeld, 1989). Furthermore, this activity took place in small group setting, where learning was cooperative. The primary aim of the activity was to increase teachers' mathematical knowledge for teaching, which has been shown to positively affect student achievement (Hill, Rowan & Ball 2005; Greenberg & Walsh, 2008; National Mathematics Advisory Panel, 2008). Our research question was: In what ways, if at all, do students' reliance on the FTA change in association with the teaching intervention?

Methodology

Our research study took place at a public, Ph.D. degree granting institution in the Northwest US during the spring of 2016. The teaching intervention was implemented in a three-credit mathematics course for K-8 teachers. All the students in the class agreed to participate anonymously in the activity, a total of 17 students. We used an intervention that consisted of four worksheets that students discussed in small groups. Previous to the investigation, students created the sieve of Eratosthenes to find the primes from 1–120 and constructed the factor tree for some numbers. They also found all the factors of a number N by using trial division or checking all the primes less than or equal to $\sqrt{N}$.

The investigation consisted of a pretest, an exploration activity, and a posttest. Each student participated in the research as part of regular classroom activity through four exploration worksheets. Though each student had his or her own worksheets, researchers encouraged discussion in small groups of 2-3 students. Also, the researchers provided some guidance through questioning and prompting students to justify their work. Immediately following the intervention, a traditional lecture was delivered which included the statement of the FTA and an example of its implications for finding factors. After this, each student worked on the posttest.

Table 1

Exploration Worksheet 1

Question	Prompt
Q1.1	Fill in each blank with the correct counting number to make each equation true. If no such counting number exists, write "IMPOSSIBLE" next to the equation. a. $2 \times \_\_\_\_\_\_ = 42$ b. $3 \times \_\_\_\_\_\_ = 42$ c. $5 \times \_\_\_\_\_\_ = 42$ d. $7 \times \_\_\_\_\_\_ = 42$ e. $11 \times \_\_\_\_\_\_ = 42$
Q1.2	Write the number 42 as a product of prime numbers. That is, find a *prime factorization* of 42.
Q1.3	Is it possible to find a different prime factorization for 42? Explain.
Q1.4	How do your answers to #2 and #3 help explain your answers in #1?

The four worksheets were used for the teaching intervention. The first three worksheets were nearly identical but each investigated a different number: 42, 45 and 88. The fourth and last worksheet provided students with an opportunity for reflection. Table 1 lists the questions used for the worksheets 1-3. Worksheet 4 consisted of a reflective question where students needed to find all the factors of a number N using what they had learned in the previous worksheets, explaining their reasoning. Table 2 shows Worksheet 4.

Table 2

Exploration Worksheet 4

Question	Prompt
Q4.1	Given that *N* is an integer with the following prime factorization: $N = 11^2 \times 7 \times 5$. Use what you have learned to find **all** the factors of N. Show your work.

The effects of the intervention were measured qualitatively with a pretest and a posttest. The questions used for these assessments are shown in Table 3. Table 4 shows the rubric used for question P1 of the pretest and posttest. The purpose of this rubric is to identify if each student made use of the FTA to decide if a number is or is not a factor of a given number N.

For example, a student who makes use of the FTA might recognize that a number would be a factor if its prime factorization were a subset of the prime factorization of the given number N. A correct use of the FTA was recorded in category C, and a partial use was recorded in category E. Other methods are merged in categories A and B: calculating N by multiplying all the factors, using trial division, factor tree and divisibility rules. We also included categories for wrong answer with or without explanations and right answers without explanations or wrong reasoning.

Table 3

Pretest and Posttest Questions

<table>
<tr><th>Question</th><th>Prompt</th></tr>
<tr><td>P1[a]</td><td>Consider the number $N = 3^2 \cdot 5^4 \cdot 11 \cdot 17^3$. Which of the numbers in the table below are factors of <i>N</i>? Explain how you know for each. If you use a calculator show the calculations you made.
<table>
<tr><th>Number</th><th>Factor of <i>N</i>?</th><th>How I know.</th></tr>
<tr><td>5</td><td>YES/NO</td><td></td></tr>
<tr><td>19</td><td>YES/NO</td><td></td></tr>
<tr><td>15</td><td>YES/NO</td><td></td></tr>
<tr><td>21</td><td>YES/NO</td><td></td></tr>
<tr><td>75</td><td>YES/NO</td><td></td></tr>
</table></td></tr>
<tr><td>P2[b]</td><td>Consider the number $867 = 17^2 \cdot 3$. Find all factors of 867 in the most efficient method that you know. Show your work. If you use a calculator show the calculations you made.</td></tr>
</table>

[a] For the posttest we used $N = 2^3 \cdot 5^4 \cdot 7^2 \cdot 13$ and the numbers on the table were: $11, 7, 14, 21$ and 98.

[b] For the posttest we used the number 637.

Table 4

Rubric for P1 (Pretest and Posttest)

Category	Description
A	Students multiply the factors and calculate the number, for instance 303,991,875. Then applied a divisibility rule.
B	Students used trial division or used the fact that a previous number (where trial division was used) was a factor. Multiply the factors and calculate the number N, for instance 303,991,875. Then divided N by the number to determine if the result was an integer number. If it has a reminder different than 0, then the number is a factor of N.
C	Student reason that if the number is prime, then it will be a factor if it is in the prime factorization of N. If the number is composite, then it will be a factor if the prime factorization is a subset of the prime factorization of N. Did not try any of the other options.
D	Student gives the correct answer without explaining their reasoning or with a wrong reasoning.
E	Student used some concepts of prime factorization or combination of methods that include an aspect of the FTA. Students under this category may have one of the following misconceptions for composite numbers: A number is a factor if the primes in the factorization are in the prime factorization of N no matter the exponents. A number is a factor if one of its primes in the prime factorization is in the prime factorization of N.
F	Wrong answer with or without explanation.

Table 5 shows the rubric used for question P2, which includes three variables that were assessed: (I) the number of factors identified of the given number N, (II) record of methodologies partially used and (III) if the student used correctly and exclusively the FTA. The second variable tells us if a student incorporated partially the methodologies described in Table 5. The use of a methodology does not necessarily indicate a full understanding of the method. For example, students that use the prime factorization tool to find factors may use the methodology by uniquely considering numbers identified in the prime factorization but then use their calculator to divide the number by each of the "candidates" to corroborate their understanding.

The third variable allows us to identify which of the students that applied the FTA exclusively and used the FTA correctly.

Table 5

Rubric for P2 (Pretest and Posttest)

<table>
<tr><th>Variable</th><th>Description</th></tr>
<tr><td>I</td><td>Number of factors reported. We took one "point" off for wrong generalizations, for example: the factors are 1,3, 17 and all the multiples.</td></tr>
<tr><td>II</td><td>Record of methodologies partially used:
<table>
<tr><th>Category</th><th>Description</th></tr>
<tr><td>W</td><td>Use divisibility rules to find some of the factors</td></tr>
<tr><td>X</td><td>Calculate the sqrt(N) and/or used trial multiplication factors less than sqrt(N)</td></tr>
<tr><td>Y</td><td>Construct a factor tree or factor lattice, and then find the factors using the tree</td></tr>
<tr><td>Z</td><td>Use prime factorization tools to find the factors</td></tr>
</table></td></tr>
<tr><td>III</td><td>Did the student evidently use a correct reasoning of the Fundamental Theorem of Arithmetic to find the factors? This is a binomial variable that can be answered with yes (1) of no (0).</td></tr>
</table>

Results

The results show that students tended to rely on the FTA more after the intervention. While the inclusion of a statement of the FTA and an example of its use as part of the intervention might draw into question the source of the effect, the literature has demonstrated that telling preservice teachers about the FTA fails to result in an appreciation of its utility in predicting a number's factor (i.e., Zazkis, 1996).

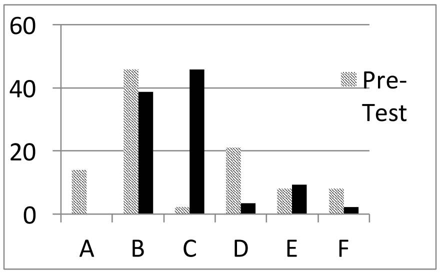
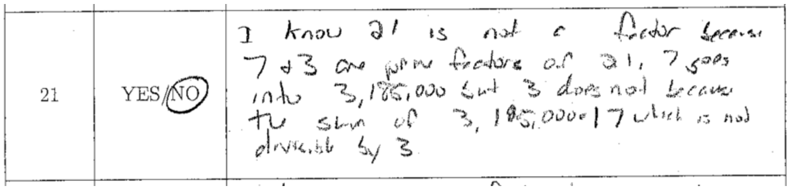


Figure 1. Classification of the methods used for each item in P1. Categories A, B, C, D, E or F are described in Table 4.

Figure 1 shows the increments of the total items in P1 that were answered using the FTA. In the pretest 2.4% of the items of question P1 were answered using a correct application of the FTA (Category C) and 8.2% using some concepts related to the FTA (Category E). On the other hand, 45.9% of the posttest items of question P1 were answered using a correct application of the FTA (Category C) and 9.4% using some concepts related to the FTA (Category E).



21	YES/[✓ NO]	I know 21 is not a factor because 7 & 3 are prime factors of 21, 7 goes into 3,185,000 but 3 does not because the sum of 3,185,000 = 17 which is not divisible by 3

Figure 2. Work of student S15 in the posttest for question P1. An example of Category E described in Table 4.

Figure 2 presents an example of Category E for the posttest question P1. In this example, a student used concepts related to the FTA but also relied on other tools to find if 21 was a factor of $N = 2^3 \cdot 5^4 \cdot 7^2 \cdot 13$. The reasoning used by the student S15 shows that the student used the FTA to tell that 21 will be a factor of N, if 7 and 3 were also factors. However, the student used a divisibility rule to determine if 3 is a factor of $N = 2^3 \cdot 5^4 \cdot 7^2 \cdot 13$ and just mentions that "7 goes into 3185000."

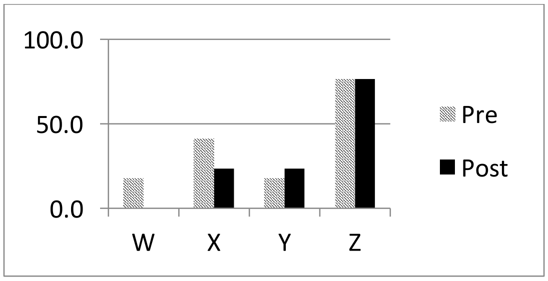


Figure 3. Classification of the methods used in the question P2. Categories W, X, Y and Z are described in Table 5.

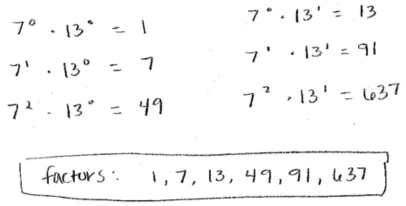
For question P2 the numbers 867 and 637 used on the pre- and post- test had 6 factors in total and students were able to find an average of 4.1 and 5.7 factors during the pretest and the posttest, respectively. Notably, 6 of 17 students were able to find all the factors during the pretest vs. 15 of 17 in the posttest. An analysis of methods used by students for finding the factors was conducted. Figure 3 shows that the same number of students used some concepts of the FTA, Category Z, to find the factors during the pretest vs. the posttest in question P2. Additionally, for variable III in Table 5, we found that none of the students used only and correctly the FTA for question P2 during the pretest versus 41.2% in the posttest.

$847 \cdot 1$
$289 \cdot 3$

$\sqrt{867} = 29.44$

Figure 4. Work of student S17 in the pretest for question P2. This was classified in categories X and Z.

The results suggest that students in the course were willing to use the FTA to find factors, but tended to support the use of the FTA with other techniques. Even when the number of students using some FTA was equal in the pretest and the posttest, students in the posttest used fewer alternative methods. Consider the work of student S17 as shown in Figure 4 and Figure 5. Here we see an example of a student that used various methods during the pretest, but only the FTA to find factors during the posttest for question P2.



$7^0 \cdot 13^0 = 1$ $7^0 \cdot 13^1 = 13$

$7^1 \cdot 13^0 = 7$ $7^1 \cdot 13^1 = 91$

$7^2 \cdot 13^0 = 49$ $7^2 \cdot 13^1 = 637$

factors: 1, 7, 13, 49, 91, 637

Figure 5. Work of student S17 in the posttest for question P2. This was classified in category Z.

Reliability

The researchers examined a sample of five pretest and five posttests to create a draft of the rubrics presented in Table 4 and Table 5. Then the researchers analyzed the collected data from the pretest and the posttest using this researcher-developed rubric. Both researchers independently scored the collected data and the scoring demonstrated 84.76% agreement. Disagreements in the scoring were resolved via debate and rubric clarification until 100% agreement was attained.

Discussion

Preservice elementary teachers that participated in the study used the FTA with more confidence after the intervention than before the intervention. Evidence of this conclusion can be found in the use of fewer alternative methods such as divisibility rules and trial division in the posttest versus the pretest. We also found that 2.4% of the students correctly used the FTA during the pretest to identify factors of a given number versus 45.9% in the posttest. These facts answered our research question: In what ways, if at all, do students reliance on the FTA change in association with the teaching intervention? Although we included all the students in our class, the size of our experiment was small.

This study is important because it provides tools for supporting a deep understanding of the FTA, which can help preservice teachers and their students to make connections for finding factors, the greatest common factor and the least common multiple of a whole number. Future questions to investigate preservice elementary teachers' understanding of the FTA are: (1) what interventions might transition students towards reliance of the FTA more quickly?; (2) can students reverse their understanding of the FTA to create a number that have certain factors?; and (3) can students create a deeper understanding of the greatest common factor and the least common multiple using the FTA?

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IMPACTING ELEMENTARY IN-SERVICE MATHEMATICS TEACHERS CONTENT KNOWLEDGE

Georgia A Cobbs
University of Montana
georgia.cobbs@umontana.edu

Gregory Chamblee
Georgia Southern University
gchamblee@georgiasouthern.edu

This paper describes how two United States Department of Education Mathematics and Science Projects impacted in-service elementary mathematics teachers' number and operations content knowledge using differing delivery models (face to face and blended). Pre and posttest data analysis found both projects significantly impacted elementary mathematics teachers' number and operations content knowledge. Delivery model design and implementation successes and challenges will be discussed. Implications for in-service and preservice content and pedagogy courses and delivery models are noted.

Introduction

The United States Department of Education Mathematics and Science Partnership (MSP) program is:

> A federal formula grant program that funds collaborative partnerships between science, technology, engineering, and mathematics (STEM) departments at institutions of higher education (IHEs), and high-need school districts. These partnerships provide intensive, content-rich professional development to teachers and other educators, with the goal of improving classroom instruction and ultimately, student achievement in math and science. Currently, funds are distributed through a formula grant to states which then hold their own competitions to award project funding. (United Stated Department of Education, n. d.)

Three MSP grants are discussed that were awarded to the University of Montana, Montana State University, and Georgia Southern University. The overarching goal of each grant is to improve the content knowledge of in-service elementary teachers. The research question addressed in this study is: Using two distinctly different delivery models, how did the mathematics content delivery within each project impact the content knowledge of practicing K-5 mathematics teachers?

Effective Professional Learning for Number and Operations

The National Council of Teachers of Mathematics *Professional Standards for Teaching Mathematics* states:

> Teachers' comfort with, and confidence in, their own knowledge of mathematics affects both what they teach and how they teach it. Their conceptions of mathematics shape their choice of worthwhile mathematical tasks, the kinds of learning environments they create, and the discourse in their classrooms. (1991, p.132)

The same document posits “[university] faculty should collaborate with other practicing professionals to design preservice and continuing education programs that reflect the issues of reform and change that must be implemented” (p.184). Today, national policy recommendation reports such as *The Mathematical Education of Teachers II* (MET II) agree:

> high-quality mathematical education of teachers is the responsibility of institutions of higher education, professional societies, accrediting organizations, and school districts, as well as PreK-12 teachers themselves. Their collective goal needs to be continual improvement in the preparation and further education of mathematics teachers. (Conference Board of the Mathematical Sciences, 2012, p.3)

Professional development must be grounded in the content needs of teachers and built on what constitutes effective professional development (PD). MET II specifically notes “for practicing K-12 teachers, content-based professional development offered by Math Science Partnerships has changed their attitudes about mathematics, and increased their mathematical interest and abilities. Moreover, it has increased the achievement of their students” (2012, p. xii). Among the MET II recommendations for numbers and operations course content are counting and cardinality, operations and algebraic thinking, number and operations in base 10, and number and operations involving fractions. The projects described in this paper incorporated these content themes into their curriculum for teachers. Furthermore, DeMonte (2013) and Reisener (2013) suggest that effective professional development encompasses the following: (1) aligns with school goals, state and district standards and assessments, and other professional-learning activities; (2) focuses on core content and modeling of teaching strategies for the content; (3) includes opportunities for active learning of new teaching strategies; (4) provides the chance for teachers to collaborate; and (5) includes follow-up and continuous feedback. The professional development models used in the two projects described here incorporated all of these components.

Methodology

With the intent to improve the content knowledge of elementary school teachers, both projects set out to develop cohesive professional development to accomplish this goal. Therefore, each project systematically developed unique approaches while incorporating the suggestive PD components mentioned above. Due to the vast geography of Montana, PD was delivered using both face-to-face and online PD, while in Georgia, only face-to-face delivery was used. However, researchers on each project were still interested in the impact of the respective PD.

Researchers on both projects created pre and posttests focused on content that was based on state standards and project course goals.

Montana STREAM Project

The Co-PIs of the Montana STREAM project wanted to have equal representation of higher education and K-8 practitioners on the design teams. Therefore, these teams were comprised of experienced mathematics teachers and university educators. The task of each team was to write online modules with complimentary face-to-face workshop sessions for K-8 teachers statewide. Initially, STREAM targeted only Grade 4-7 teachers because in Montana teachers receive a K-8 teaching license and sometimes lack the mathematics content necessary in the upper grades. The project materials addressed a series of themes aligned with Common Core mathematics content domains (e.g. number systems and operations, ratio and proportions, data and statistics, geometry). In Year 1, the STREAM partnership included teachers from 15 Montana school districts. Then in Year 2, additional teachers from five districts participated. Even though 85 teachers were involved during the first two years, matching up the pretests and posttests resulted in only 39 participants for Year 1 and 19 participants for Year 2. Both cohorts experienced approximately 140 hours of PD online and face-to-face. A pretest was the initial task the participating teachers engaged in on the first day of their professional development. The total possible score was 47. Test items included items involving numbers and operations (e.g. place value and computation), and others similar to the following:

1. If you buy something at a 40% discount off the listed price of $68, what price do you pay? Place your answer in the box provided.
2. In a certain company, the ratio of the number of female employees to the number of male employees is exactly 3 to 4. Which of the following could be the total number of employees in the company?
   [A] 81 [B] 87 [C] 91 [D] 95 [E] 101
3. Jesse runs 10 meters in the same time as Owen runs 8 meters. One day they ran around a 400-meter circular track. They started at the same place at the same time and ran in opposite directions. What was Jesse's location on the track when Owen passed the starting line the third time? Place your answer in the box provided.

Professional development in Year 1 was delivered over six months beginning with a face-to-face workshop (8 hours), including introductory activities from each of the four modules (Mathematical Practices (MP), Number Systems and Operations (NSO), Fraction, Ratio and Proportions (FRP), and Teacher Learning and Leadership (TLL)). Teachers then participated in these modules via asynchronous 4-week online delivery (24 hours each). Partner districts were located across Montana, warranting the need for online learning, which proved to be successful most of the time, with only a few minor issues with connectivity. Finally, a four-day summer academy (30 hours) took place with all teacher cohorts from each district. Teachers received final information from each module and they then developed a strategic plan to share professional development within their schools. These plans, accompanied by a small budget, were implemented throughout the following school year.

The STREAM leadership team developed guidelines for all online modules including tasks that were both individual and group oriented. Besides teaching content and pedagogy, the modules were to be completed with 6-8 hours of teacher time per week, keeping in mind the busy work load of teachers. Oftentimes teachers were able to try an activity with their own students and provide student data and feedback. The NSO Module incorporated these guidelines with the following individual and group activities:

Activity 1. Community Building: Numbers About Me
Activity 2. Content: Characteristics of a Learning Progression
Activity 3. Create a Learning Progression (Group)
Activity 4. Align Mathematical Tasks to the Montana Common Core Standards
Activity 5. Read a Professional Article (with group discussion following)
Activity 6. Write a Learning Plan

Time frames changed a bit from one year to the next. While the Year 1 cohort's first year was only five months, the Year 2 cohort followed a similar pattern of events, but spanning eight months, with an additional face-to-face mid-year workshop. Also, two new modules were created: Geometric Thinking and Data and Statistics. Modules were offered on a rotating basis, keeping teacher requests in mind. The Mathematical Practices continued to be the most requested modules. The posttest was given to both sets of cohorts during the Summer Academy. The results are discussed below in the findings section.

Georgia Southern Project

The Georgia Southern University project served nine rural school districts in the same Regional Educational Service Agency (RESA) office in central Georgia. Two university faculty taught in the project. One faculty member was a mathematics educator in the College of Education with a second faculty member from the mathematics department. Faculty members co-planned and co-taught classes. Participant selection was based on principal recommendation. Teachers from 15 schools enrolled in the project. Project design was based on Georgia's K-5 mathematics specialist endorsement requirements in four content areas: Number and Operations, Data Analysis and Probability, Geometry and Measurement, and Algebra. For the purposes of this paper, only Number and Operations content mastery will be discussed. The Number and Operations course was 60 face-to-face contact hours with approximately 20 hours of instructional time dedicated to overarching pedagogy concepts for teaching mathematics (e.g. defining problem solving, assessment strategies, teaching all children mathematics) and 40 hours to Numbers and Operations content. Content delivery consisted of five instructional days during the academic year and a two-week summer institute. Thirty-one teachers attended the initial information session and completed a pretest. Twenty-three (23) teachers completed the Numbers and Operations course and the posttest, resulting in 23 matched pairs. Teachers' content knowledge was pre-and posttested using a project developed and state approved Number and Operations test. Each test consisted of 14 identical content items testing fraction, decimal and percent conversions; place value; base 10 representations; alternative (non-traditional) operations algorithms; inverse operations; and misconceptions. Possible scores ranged from 0-100 points. An item by item correct versus incorrect percentage test analysis was conducted to determine if there were content misconceptions for both the pretests and posttests. Test items included items involving numbers and operations (e.g. place value and computation) and others similar to the following:

1. Provide 2 different pictorial representations of the number 367 without using numerals.
2. Use an alternate algorithm to find the difference between 277 and 456 and explain it.
3. Identify if each solution is correct or incorrect. Explain why the method is correct or incorrect. $\frac{2}{3} \times 4 = \frac{8}{12}$

Findings

Analysis of the Montana project (paired t-test) found significant differences in elementary teacher's content knowledge for both cohorts. As evidenced in Table 1, gains in posttest score comparison to pretest score for each individual teacher were determined to be significant if the posttest gain was greater than one-third of the pretest standard deviation (SD): $Post>Pre+(1/3)*SD_{PRE}$ where the pretest standards deviation was equal to 6.99. Thirty-three (84.6%) of 39 teachers made significant pretest to posttest gains (Shaw, 2013). Year 2 participants used the same test, resulting in 19 matched pretest and posttest total scores. Using the same benchmark for gains, ten (55.6%) teachers made significant pretest to posttest gains in content knowledge (Shaw, 2014). See Table 1 for details. Gains in content knowledge are only one aspect of teacher growth; additional qualitative and quantitative evidence demonstrated increased knowledge of standards and acquisition of leadership skills.

Table 1

Change in Content Knowledge (Years 1 and 2)

Cohort	N	Mean	Std. Dev.	t statistic	Pval
Pretest Y1	39	30.69	6.99	7.93	$< 10^{-3}$
Posttest Y1	39	37.46	6.87		
Pretest Y2	19	33.21	4.98	2.18	0.0215
Posttest Y2	19	35.74	4.68		

The Georgia Southern University project pretest mean was 61.71 with a posttest mean of 89. A Wilcoxon Signed-Rank test found a significant difference in pretest and posttest scores ($p<0.01$). Data analysis using the MSP-required MSPTCK test found significant pretest to posttest gains for all 23 teachers. Pretest misconceptions were focused in the following three areas: (1) fraction, decimal, percent conversions especially in the areas of repeating decimals versus terminating decimals (e.g. ⅔ is not the same decimal, fraction and percent as 66%) [N=21]; (2) non-traditional algorithm solution explanations to problems using the four basic operations (e.g. estimation [N=9], mental math [N=20]); and (3) finding and correcting students' misconception errors on specific tasks (e.g. division using a non-traditional algorithm [N=19]). Posttest data analysis found teachers still had difficulty with (1) fraction, decimal, percent conversions especially in the areas of repeating decimals versus terminating decimals (e.g. ⅔ is not the same decimal fraction and percent as 66%) [N=18], and finding and correcting students' misconception errors on specific tasks (e.g. division using a non-traditional algorithm [N=6]). Overall, teacher content knowledge increased as a result of the project. However, teachers continued to have several misconceptions about mathematics at the end of the course.

In general, both models accomplished their goal of positively impacting teacher content knowledge. The mode of delivery did not impede content knowledge growth.

Recommendations

The Montana STREAM and Georgia Southern University projects support recommendations found in the NCTM *Professional Standards* and CBMS MET II that in-service teachers need continued professional development in mathematics content. Both projects confirmed this need by documenting that some participating teachers lacked content knowledge of number and operations. Both projects helped these elementary teachers gain content and pedagogical knowledge through learning and collaborating with colleagues. In-service teacher misconceptions imply preservice content and pedagogy courses need to continue to focus on more than procedural fluency. For example, in-service teachers continue to struggle with conceptual understanding with non-traditional algorithms. Both projects implemented effective professional development criteria and were successful using these components: content-driven; unfolded over an extended period of time; integrated research-based teaching strategies and active learning; and encouraged teachers to collaborate and continue interacting with each other beyond the project. Both projects support the premise that professional development offered through differing delivery models can impact in-service teacher content learning. Both models had similar and different implementation challenges. Similar challenges were recruitment and retention (time commitment issues) over the course of each project. The face-to-face project (Georgia Southern) encountered attendance issues due to lack of substitutes during the academic year, teachers having to commit to two weeks of summer instruction and teachers completing all endorsement assignments in a timely manner while working full-time. The blended model (Montana) had some teacher turnover as well as administration changes. Both impacted the level of implementation of the teachers' strategic plans. Overall, these two projects document that mathematics teacher learning is a career-long process. Using either modality of delivery, the professional learning is most effective when based in mathematics content. Developing in-service elementary mathematics teacher content knowledge continues to be an area of need and a rich environment for research.

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DESIGNING PROFESSIONAL DEVELOPMENT MODULES FOR TEACHER LEARNING AND PRACTICE

Seanyelle Yagi
University of Hawai‘i
slyagi@hawaii.edu

Fay Zenigami
University of Hawai‘i
zenigami@hawaii.edu

Linda Venenciano
University of Hawai‘i
lhirashi@hawaii.edu

This paper reports initial findings about teachers' interactions with an online professional development module designed to enhance their understanding of measurement concepts foundational to the number line through experiences with continuous quantities. Designing an online interface, which takes a problem-solving approach while at the same time, provides guidance for teachers was non-trivial and led us to consider design features available through the technology. The results suggest teachers gained insight into the number line and highlighted the inclusion of human interaction as valuable for their learning, important considerations informing the module's continued development.

Theoretical Framework and Related Literature

One of the few formal activities in which teachers interact with each other and learn new content and pedagogy is in professional development (PD). Effective PD is generally agreed to include the following features: (a) content focus, (b) active learning, (c) coherence with other expectations and goals, (d) sustained duration, and (e) collective participation of teachers from a school, district, and/or grade level. However, relatively few teachers appear to be offered professional development opportunities that can be characterized as meeting these criteria in a meaningful way (Desimone, 2009; Desimone et al., 2002; Garet et al., 2001; Hill, 2004).

In-person PD is valuable for building relationships and content-focused collaborations, but online PD affords teachers additional opportunities to engage in collaborative, work-embedded PD, as well as to expand their interpersonal network. Online PD is flexible in scheduling, and avails sources of content and pedagogy beyond the local context. Online educational resources are already widely available. Teachers access popular, free sites such as Teachers Pay Teachers and Pinterest to acquire lessons or Khan Academy to review mathematics content. The problem is that while options like these are accessible, they are not designed to support development of high-quality teaching. Online PD has the potential to provide meaningful support to teachers, particularly when that PD environment is easy to access and meets the needs of teachers in locations that do not have in-person access to valuable resources. Although the popularity of online PD resources and teacher networks is growing rapidly, little is known about emerging cloud-based computing practices for teacher PD. More research is needed to identify best practices for the design and implementation of online teacher PD models (Dede et al., 2009).

Study into the human experience in the workplace with the integration of technology began in the late 1950s. Through these experiences, there emerged the notion that technology should not be the controlling factor in determining the workplace environment for employees, a construct central to the socio-technical theory (Mumford, 2006). The socio-technical theory describes the need for “the joint optimization of social and technical systems” (Mumford, 2006 p. 321), where human needs are not forgotten when technology is introduced, and the social and technical are equally weighted in the workplace. Within the context of designing and delivering online teacher PD, it is important to consider humanistic needs when using technology as a platform for teacher learning.

The PD Module

To meet the complex needs of teachers within their local contexts and provide them with meaningful learning experiences, a team of mathematics educators at the Curriculum Research & Development Group developed an online PD module, *A Little Professional Development about Big Ideas (ALPBI): Measurement Foundations to the Number Line*. Based on a series of lessons from the Measure Up (MU) curriculum research and development project (Dougherty, 2008), the module includes four investigations focusing on measurement concepts foundational to the number line. In US elementary curricula, measurement is typically treated as an isolated topic, and measurement tools are used in rote fashion, often without understanding of the concepts that they embody (Clements & Battista, 1992). MU is based on an innovative approach developed by a team of Russian mathematicians, educators, and psychologists led by Davydov and El’konin, and grew out of Vygotsky’s cultural-historical theory. MU uses measurement as the context to study foundational concepts in mathematics (Venenciano & Dougherty, 2014), and requires the use of continuous quantities, such as area, volume, mass, and length. MU begins with a non-numeric approach using generalized measurement concepts to develop students’ quantitative and numeric reasoning. Although virtual manipulatives that mimic real world tools, such as standard rulers and calibrated beakers, exist to support the teaching of measurement concepts, these often do not provide teachers and students with opportunities to explore and develop understandings of the concepts that underlie these tools and their calibrations. We adapted a series of MU lessons for teachers to engage in measuring of continuous quantities and inquire about concepts, like unit and the number line, through the investigations.

In addition to the innovative mathematical content, the module was designed with educational and design features to highlight pedagogical considerations and enhance the PD experience. The module was designed to capture the inquiry and problem solving processes that would otherwise be modeled by facilitators to engage participants in a face-to-face session. To this end, the design goals were to 1) engage teachers in an inquiry approach; 2) encourage teachers to actively do mathematics; 3) provide opportunities for reflection; and 4) present mathematics via problem solving. To address these goals, we turned to interactive features afforded by online, cloud-based technology, Google docs. We included open-ended question prompts and hyperlinks to other documents, which provided insight into each topic under discussion. Hyperlinks also linked icons and words within the module to other docs in which content and instructional considerations (e.g. concurrent representations) were highlighted. A reader-friendly, conversational text also guided the reader through the module.

In this early phase of our research, we examined how teachers interacted with the educational and design features to inform the development of the module. Our research questions are: (a) How does the ALPBI design support teachers' mathematical thinking about the number line? and (b) How do teachers interpret ALPBI as PD? Teachers' interactions with our module will advance understanding of the teaching and learning of mathematics in technologically rich PD environments and provide information for continued development of the module. There is inadequate research available for PD planners seeking best practices for the design and implementation of online teacher PD models (Dede et al., 2009; Prestridge & Tondeur, 2015). Our project, drawing from the socio-technical model (Ropohl, 1999), seeks to address these influential factors and tensions in relation to the online PD design.

Methodology

Seven elementary classroom and resource teachers from six different schools were recruited to individually read through and complete the investigations in the ALPBI module prototype within a two-week period. All of the teachers were experienced with teaching elementary mathematics; three were classroom teachers, and four served as school or state-level teacher leaders who provided coaching or instructional support for teachers. Initial communication with the teachers was through email, with no face-to-face orientation prior to their interactions with the module. This was intentionally done to see how teachers would work through the module with minimal direction and guidance. They were provided with specialized materials (e.g., spring scales) that might not be typically found in classrooms, but were expected to find everyday materials needed to complete the tasks. Teachers were then emailed the link to the module and given two weeks to work through its four investigations.

Prior to working on the module prototype, the teachers completed an initial survey, which asked them about their current understanding of the number line and how they saw it used in instruction. Following completion of the ALPBI module, the teachers completed a post-survey that contained items in which they could rate parts of the module and their perceived changes in their understanding of the number line on a Likert scale. On the post-survey teachers were also asked to provide a brief written response about the overall value of the module as an option for PD. Additionally, the teachers were placed in two groups according to their availability and each group participated in a 1-2 hour focus group meeting with the research team. Since this was the first face-to-face meeting with the teachers, some time was spent providing the purpose and background of the module and the project. Using their post-survey responses as a catalyst, teachers discussed their thoughts and experiences with the module. The focus group meetings were audio-recorded and transcribed, and researcher field notes were taken during the meeting. Both the audio-recording transcripts and the notes were reviewed to provide supporting evidence and further insight into the survey responses. Analysis of the data was done after the focus group meetings were completed. The qualitative responses from the pre-survey were coded and analyzed collectively followed by the coding of the post-survey responses according to their analytical properties (Charmaz, 2006). The codes were developed inductively from the data and categorized into themes by two researchers who first coded survey responses independently and then compared their analyses. The researchers discussed discrepancies in the coding until a resolution was reached. Post-survey ratings were used for contextual understanding when interpreting qualitative data from open-ended items and focus group discussions.

Findings

Results from the pre-survey

The teachers responded to two questions on the pre-survey: 1) What is a number line? and 2) From your knowledge and experiences, how is the number line used in mathematics instruction? The vocabulary and mathematical ideas that were included in the responses indicated teachers' notions of the number line and their thoughts on the use of the number line during instruction. The findings suggest a focus on the number line as an object or tool, with its physical features highlighted (e.g. a line marked off with numbers). Three teachers discussed the number line's relationship to linear measurement, including comments such as, "A number line is a length model, each number represented by its length from zero" and "It could be a precursor to measuring lengths." One teacher described the number line as "a continuum with numbers," while another wrote that it was "a representation of continuous numbers." Since neither clarified what was meant by "continuous" or "continuum," the researchers are uncertain about the teachers' understanding of the number line in this respect. In terms of classroom instruction, teachers saw that the number line could be used as a tool for counting or computing, for measurement, and to highlight number relationships.

Results from the post-survey and focus groups

The researchers met with the teachers in two focus groups, according to their availability and location. One group consisted of three teacher leaders, while the other group included one teacher leader and three classroom teachers. Two of the classroom teachers implemented the investigations as lessons with their first-grade students. They described initially feeling frustrated, however, later came to view the experience as a means for deepening their understanding of the concepts in the module. One teacher shared, "I really like that, being able to visualize and then you see it hands on. You're actually doing it, pouring it, so they can actually see it. Whereas if they just see the number line, you tell them, okay count up or whatever."

All seven teachers completed the post-survey and reported that they had read at least 90% of the module text. Teachers rated their perceived change in understanding of the number line on a Likert scale of 5 (*significant improvement*), 4 (*some*), 3 (*unsure*), 2 (*very little*), and 1 (*none at all*). All but one teacher reported significant or some improvement in their understanding of the number line as a result of using the module. In describing the significant improvement, the teacher wrote that the module "deepened my understanding of the number line beyond the basic add/subtract … I never thought of [measurement] as an abstract concept … when dealing with representations of length, volume, and weight."

Teachers' perceptions of the overall value of the module as an option for professional development are characterized by three themes: 1) Self-improvement/Learning; 2) Instructional Strategies/Teaching; and 3) PD delivery. Overall the teachers found completing the PD module as valuable for their professional learning. Several responses highlighted instructional insights afforded as a valuable part of their interactions with the module (e.g., "This module definitely gave me a different and innovative approach to teaching the number line," and "I would have loved to see potential misconceptions so that could be addressed by pre-planning investigations and questions to ask students during the debrief sessions."). Furthermore, two teacher leaders discussed aspects of the PD experience thought to be important for teacher learning and success with the module. One commented, "I think that this module is a great option for professional development... the module would be best done if someone were facilitating the modules through a data teams (or else having teachers do the module together in a small group, if a facilitator is not an option)." Another teacher leader highlighted the value of collaboration, remarking, "It might be better for teachers to complete this module with a partner. It would be great to share insights and/or wonderings with someone regarding the investigations." Other ideas for improving the module included the use of visual images, (e.g. classroom videos, drawings) to facilitate teachers' understanding of the pedagogical considerations discussed and modeled by the module. One teacher stated, "What if we take video of a lesson like this and use that as part of the PD? So that teachers can see that's what this lesson is supposed to look like, or that's the kind of responses you might be able to have." Another teacher added her own illustrations on her copies of the module, "because I had to understand what was going on, what was required."

Discussion and Implications

As online teacher PD continues to grow in popularity, more research is needed about the design and implementation of online PD models that provide contexts for teachers' meaningful interactions with new content and pedagogy (Dede et al., 2009). Our approach to online PD combined innovative mathematical content with design features made available by the technology that modeled instructional approaches consistent with inquiry and problem solving. Teachers found the experience of working through the tasks valuable in advancing their understanding of the number line, and interactions with others (e.g. with a partner/facilitator) as important for enhancing their learning while they worked through the online interface. Although online PD affords a myriad of opportunities for professional learning, preliminary findings from this study suggest the importance of in-person discussions and active engagement with the mathematical tasks in conjunction with the online experience. Our findings suggest incorporating social interaction with affordances of the technology to enhance teacher learning under both social and technical conditions (Ropohl, 1999). In addition, two teachers found the experience of implementing some of the lessons with their own students challenging, causing them to raise questions such as "What is the thinking behind this?" and "What do I want them (students) to get out of this?," but at the same time valuable for their own professional learning. This supports the notion of teacher development occurring not only through structured PD, but also in other professional experiences as well, including within their practice (Borko, 2004; Desimone, 2009).

Finally, although our research intent was not to compare differences in teachers' interactions, we noted qualitative differences in the conversations depending on teacher roles. The classroom-based teachers tended to focus on instructional insights afforded them at a personal level through interacting with the module, whereas the resource teachers focused on teacher PD at a general level and tended to address the delivery of the module. The results of this study present several considerations we, as curriculum developers, will take into account as we continue the module's development.

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TRAINING GRADUATE STUDENT INSTRUCTORS AS PEER MENTORS: HOW WERE MENTORS' VIEWS OF TEACHING AND LEARNING AFFECTED?

Sean Yee
University of South Carolina
yee@math.sc.edu

Kimberly Rogers
Bowling Green State University
kcroger@bgsu.edu

To develop graduate student instructors' (GSIs) skills and abilities as collegiate mathematics instructors, researchers at two universities implemented a peer-mentorship model where experienced GSIs completed a 15-week professional development (PD) to learn how to mentor novice GSIs in teaching undergraduate mathematics. Using pre-survey, post-survey, and semi-structured reflective interviews, we studied changes in eleven mentor GSIs' perspectives on teaching and learning practices and what aspects of the mentor PD were deemed valuable by the mentors. Results suggest that this mentor PD, as a peer-mentorship model, helped GSIs deconstruct the dichotic mathematical paradigm of statements being true or false when discussing teaching.

A key ingredient in a successful collegiate mathematics department is the "effective training of graduate teaching assistants" (Bressoud, Mesa, & Rasmussen, 2015, p. 117). This training is crucial because mathematics graduate student instructors (GSIs)[1] serve as instructors of record for hundreds of thousands of undergraduate mathematics students each semester (Belnap & Allred, 2009) and significantly impact the quality of collegiate mathematics instruction across the United States of America. Despite their prevalent role as instructors of undergraduate mathematics, GSIs typically lack guidance and support to teach undergraduate students effectively (Speer & Murphy, 2009). In order to address this critical need for early support in GSIs' development as effective teachers, this study generated and implemented a mentor professional development (PD) at two American universities to develop experienced GSIs into mentors for novice GSIs (protégés). This paper explores the effect of a mentor PD[2] on mentor GSIs' views of teaching and learning.

Related Literature

GSI Guidance and Support

In K-12 teacher education, the critical role of student teaching with a mentor teacher has been recognized as a vital precursor to fully instructing a course (Council for the Accreditation of Educator Preparation, CAEP Standard 2). At the collegiate level, no such standard precursor exists across doctoral granting institutions (Speer & Murphy, 2009). This is due, in part, to the wide variety of roles graduate students may be assigned, such as tutoring, grading, recitation instructors, or instructors of record. Moreover, the varied and limited resources within mathematics departments often present unique challenges to providing GSI support.

Although mathematics faculty members and course coordinators may offer general advice about teaching to graduate students, rarely is this advice individualized enough for the GSI to justify and reflect on their pedagogical decisions (Speer et al., 2014). This is where a mentor can be helpful by offering specific advice and justifying that advice by articulating pedagogical decisions. Many universities have used faculty as mentors for GSIs, however Johnson and Nelson (1999) found that such relationships are ethically complicated and multifaceted because of other hegemonic roles faculty must play, such as doctoral advisors and qualifying exam evaluators. We posit that to be genuinely aware of the individualized pedagogical decisions requires a mentor closely in tune with a protégé's current experiences. To that end, we focused on a mentor PD for experienced GSIs to guide and support protégés.

Mentoring GSIs

Research has indicated that mentoring has social and cultural benefits if mentor GSIs are focused on helping protégés learn how to teach. Johnson and Nelson (1999) indicate that mentoring is central to "quality graduate education" (p. 205), a key component of a successful mathematics department (Bressoud et al., 2015). Crisp and Cruz's (2009) meta-analysis of mentoring literature from 1990 through 2007 found that certain subgroups (minorities and females) benefited greatly from peer mentoring, as mentors offer support to socialize professionally, work, navigate, reflect on academic discourse, and help alleviate stress within doctoral programs. Zaniewski and Reinholz (2016) looked at mentoring from a cultural perspective of identity in a peer mentoring program and found similar positive psychosocial and academic interactions resulting in friendships that generated a *community of practice* (Kensington-Miller, Sneddon, & Stewart, 2014) amongst peers. Such results are desired within mathematical graduate doctoral programs. Thus, the literature supports the design and implementation of peer-mentoring for GSIs, yet raises the question: How do we mentor the mentors?

Mentoring Curricula

Although teaching experience is necessary, it is not sufficient for mentoring because mentors need to understand *their* role and purpose in facilitating meaningful pedagogical decision-making conversations with protégés (Rogers & Steele, 2016). Despite a small body of literature surrounding structuring mentor PD curricula (Crisp & Cruz, 2009), we note Boyle and Boice's (1998) seminal work on mentoring both novice faculty and novice GSIs with tenured faculty where they considered mentoring as the “cousin of faculty development” (p. 158). These researchers compared spontaneous mentoring (talk to the mentor if there are problems) and systematic mentoring (meeting regularly every week) and found that systematic mentoring was more effective in supporting GSIs and faculty because the mentor could not prepare appropriately when it was spontaneous. Boyle and Boice also observed the topics that dominated mentor meetings in decreasing order of frequency were (1) discussions of undergraduates, (2) teaching styles, (3) teaching-related goals, (4) grading issues, and (5) course preparation.

Boyle and Boice’s (1998) results informed the framework of our mentor PD because all five frequented topics could be discussed within the mentor PD through two main responsibilities: observing protégés teach and running small group protégé meetings systematically (not spontaneously). Thus, our mentor PD curriculum revolved around observing protégés (including post-observation discussions) and facilitating small group discussions. This study examines the impact of a 15-week mentor PD on experienced GSIs via the research question*:* How did the mentor PD change mentors’ perceptions of student behavior, student learning, and/or effective teaching?

Method

Participants

Experienced graduate students at two universities applied and were selected to be mentors by the researchers based on their teaching experiences (aptitude for implementing student-centered techniques), their pedagogical accolades (teaching awards and student evaluations), and their desire to help novice GSIs to improve teaching at their university (essay responses were required). The number of participants was determined by the average size of each university’s mathematics GSI program. Eleven mathematics doctoral candidates were selected to participate in the mentor PD seminars (four from one university and seven from another).

Mentor PD Curriculum

The goal of the 15-week mentor PD was to equip the eleven experienced GSIs to be effective peer-mentors. The participants and a mathematics education researcher met for 50 minutes once a week to discuss the responsibilities of the mentor as well as to generate frameworks and perspectives necessary for mentoring. Revolving around observing protégé GSIs and facilitating bi-weekly small group meetings (one mentor with four protégés), Table 1 describes the mentor PD curriculum of this study:

Table 1.

Mentor PD Curriculum Topics in Order of Discussion

Weeks	Topic	Description
1-2	Aligning lesson goals, assessments, and mathematical tasks	Reviewed teaching-related goals as well as designing and aligning lesson plans.
3-6	Designing, organizing, and implementing GSI observation protocol (GSIOP)[*]	Mentors used the GSIOP to observe other mentors' classrooms to become familiar with the GSIOP.
7-8 & 13-14	Facilitating post-observation discussion, including how to support protégé development as teachers	Discussed structures to provide constructive criticism for post-observation discussions via role-playing and video-based scenarios.
9-10	Designing, organizing, and implementing small group meetings	Discussed topics, structure, and protégé needs during protégé's first year of teaching.
11-12	Facilitating small group meeting discussions	Discussed productive discourse and analyzed small-group scenarios to limit mentor opinions
15	Critical reflection during small group meetings	Discussed critical reflection (Brookfield, 1995) to help protégés solve curricular problems.

[*] The GSI Observation Protocol (GSIOP) was modified from the MCOP2 (Zelkowski, Gleason, & Livers, 2016) and collegiate teaching observations.

Although researchers collected rich observational data from the mentor PD, space limitations mandated this paper remain focused on the results that would answer our research question.

Mentor PD Data Collection

At the beginning and end of the mentor PD, the eleven mentor GSIs answered a survey to examine their attitudes, experiences, and conceptions about teaching collegiate mathematics. We modified Jong, Hodges, Royal, & Welder's (2015) Mathematics Experiences and Conceptions Surveys to focus on the tertiary instructors. The pre- and post-surveys shared a group of questions that asked the mentors to rate how strongly they agreed with statements in three categories: (a) beliefs about students (15 statements), (b) teacher characteristics (11 statements), and (c) lesson design (3 statements). On a scale of one to five, with one being strongly disagree and five being strongly agree, participants rated their agreement with statements such as: (a) students should use multiple ways to represent concepts and solve problems (beliefs about students), (b) as a teacher I provide wait time and think time regularly (teacher characteristics), and (c) the structure of my lesson must be well organized to effectively achieve its goals (lesson design). We analyzed the mentors' pre- and post-survey responses to address our research question. After the conclusion of the mentor PD, an external evaluator conducted 1-hour, semi-structured reflective interviews with each mentor. The mentors were given a copy of their pre- and post-survey responses and asked to elaborate on what they saw in their own responses to triangulate the data. Mentors' responses about how and why their attitudes changed throughout the mentor PD helped to answer our research question.

Mentor PD Data Analysis

We first examined the results of the pre- and post-survey questions. This quantitative analysis informed the design of the semi-structured reflective mentor interviews, which we qualitatively coded relative to the mentor's responses to the attitudes, experiences, and conceptions on teaching questions. For all 29 pre and post questions, t-tests were used to determine variance before and after the mentor PD. Aggregate quantitative analyses of the pre- and post-survey data were shared during the semi-structured interviews to help answer the research question.

Results

Paired sample t-tests were implemented on all 29 pre and post survey questions to look for variance (N=11 with $alpha$=5%). Although, no variance was significant (which may be due to the limited number of participants), a few descriptive statistics on change in mean offer insight into how the mentor PD affected specific mentors' attitudes. Due to limited space, we discuss the survey question with the greatest change in attitude. Mentors attitudes that "students' success in mathematics depends primarily on how hard they work" had the largest average negative value change after the mentor PD (ΔM=-0.64, ΔSD=1.43) and was almost significant (p = 0.081). This indicates that, on average, mentors agreed less with this statement after the mentor PD.

In the reflective interviews, nine of the eleven mentors did explicitly discuss their negative change in attitude on the survey relating student success and hard work. One mentor said:

> When I first took this [pre-survey], I strongly agreed because it sounded right, the way it should be, but what caused me to change was the word 'primarily' because success in mathematics is *in part* how hard they work…but a good teacher *certainly* makes a difference, the resources available *certainly* makes a difference. If not, we are assuming students…are lazy and just don't work hard, which is not true in my opinion.

A second mentor corroborated this perspective in their interview by directly connecting his negative change in attitude to the mentor PD:

> I probably changed my answer after the seminar because I had seen poorer examples of instruction in the mentor PD, which leads me to believe that, despite what I would like, the quality of instruction plays a role in how well they learn the subject material.

Another mentor focused on deconstructing the word success:

> The hard part for me to really piece together about this question is 'student success' in mathematics. If I have a student who comes in with really strong [mathematical] background and aces all the homework, aces all the exams, versus the student who improves greatly but does not get as good of a grade, what is that in terms of success?

Altogether five of the nine mentors discussed disagreed with the word success, three of the nine disagreed with the word "primarily" (citing teaching as another important factor), and one of the nine unpacked the notion of hard work, arguing that there were differing views of hard work that can lend themselves to successful mathematical understanding. Fundamentally, during these interviews, mentors were critically reflective (Mentor PD, week 15) as they deconstructed the meaning of certain words such as "hard work", "primarily", and "success" that they took for granted prior to the mentor PD.

To investigate what aspects of the mentor PD may have contributed to this changed, we asked mentors to specify what parts of the mentor PD they considered most valuable in critically reflecting and deconstructing these terms (i.e. hard work, primarily, and success). Ten of the eleven mentors referenced facilitating post-observation discussions (Weeks 7-8 and 13-14) where mentor GSIs observed each other, analyzed video-based scenarios, and discussed written-case studies where mentors debated about strategies for identifying and communicating ways for protégés to improve. This specifically speaks to mentors appreciating the mentor PD as a means to practice productive discourse with protégés.

Discussion

In sum, our study provides valuable information about how a peer-mentorship model engaged mentor GSIs' perspectives on teaching and learning. Although the t-tests indicated no significant variance in mentors' perspectives of teaching and learning (*alpha*=5%), the reflective interviews indicated qualitatively that the mentor PD resulted in mentors thinking about certain terms, such as "success", as relative to courses and students they were currently teaching. Thus, mentors were able to deconstruct the dichotic paradigm (true versus false) prevalent in mathematical statements but not mathematics education. When mentors provide constructive feedback to protégés after observing their classroom teaching, it is crucial that they address teaching concerns with subjective understanding of words such as "success" so as not to indicate to protégé GSIs that there is an absolute correct way of teaching or defining student "success".

Implications for Research

This study contributes to the field's knowledge of GSI guidance and support by exploring how this mentor PD (part of a peer-mentorship model) can engage experienced GSIs' understanding of mentorship, teaching, and learning. Our research demonstrated that the mentor PD did promote GSIs' ability to justify their pedagogical understanding of certain terms. Their choice to value discursive facilitation and their ability to deconstruct terms such as "primarily" and "success" demonstrates they were able to consider multiple factors needed to reason through pedagogical decisions, a crucial area of concern in recent research (Rogers & Steele, 2016).

Implications for GSI Programs

The mentor GSIs' value of facilitating discourse in protégé post observations is an ideal way to begin in developing peer-mentoring programs for GSIs because from this discourse stems the foundation of a community of practice amongst GSIs within mathematics departments. In their research on how mathematics teaching practices shift undergraduate instructors' academic identities, Kensington-Miller et al. (2014) emphasize the need for a community of practices for an improvement in teaching practices to take place. These researchers define a community of practice as:

> a place of collaborative inquiry where various approaches to teaching can be tested through a reflective sharing process . . . . [A community of practice] can contribute to deeper levels of awareness and achieve new learning that can, in turn, lead to significant change. (Kensington-Miller et al., 2014, p. 829)

To work through reflective sharing for achieving awareness, a community requires a safe environment with a knowledgeable facilitator for productive discourse, which was at the center of the mentor PD and valued by the mentor GSIs. Thus our data corroborates and aligns with prior research (Boyle & Boice, 1998; Kensington-Miller et al., 2014; Smith & Stein, 2011) by underscoring the need for a mentors to be skilled at facilitating productive discourse in a collaborative environment and systematically organizing mentor meetings in hopes of generating a sustainable community of practice.

Endnotes

[1] GSI was used instead of TA (Teaching Assistant) because GSI targets the specific set of graduate students who are full instructors of record.

[2] The implementation of this mentor PD was supported by a National Science Foundation collaborative grant at two universities (NSF IUSE GRANT # 1544342 &1544346).

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PROMISE OF A SOCIOCULTURAL PERSPECTIVE OF EMOTION IN MATHEMATICS EDUCATION

Carlos Nicolas Gomez
Clemson University
Carlos@Clemson.edu

Previous research on emotions in mathematics education centered on students doing mathematics. This cognitive perspective limits emotions to only the individual and does not consider the influence of interactions with others while learning and doing mathematics. A sociocultural perspective on emotion needs to be considered to move the field forward. The sociocultural perspective on emotion creates a space where the emotions of prospective and in-service mathematics teachers, as well as classroom environments, can be investigated. In this paper, I discuss the important constructs of the sociocultural perspective on emotion and describe the perspectives potential in mathematics education research.

Becoming a teacher is a complex process influenced by internal and external factors (Borko et al., 1992). One factor researchers have focused on is the influence of emotions on mathematics teachers' professional development (Hodgen & Askew, 2007). Previously, most research on emotions centered on students doing and learning mathematics (McLeod, 1992; Philipp, 2007). This work has provided descriptions and understandings of students' emotional states as they solve mathematical problems and how emotions influence problem-solving (Goldin, 2014). Research on the influence of prospective teachers' emotional experiences on their decision-making processes, however, are few. Researchers have found the emotional experiences had during teacher preparation coursework motivates prospective teachers' decisions to use research based practices (Gomez, 2016). Mathematics educators need to consider the emotions of prospective teachers because mathematics learning elicits particular emotions from both students and teachers due to the relationship one has with the content (Williams-Johnson et al., 2008).

Research on emotions can be described as using either a cognitive or sociocultural perspective. The cognitive perspective on emotions has dominated mathematics education research (see McLeod's (1992) and Goldin's (2014) review of literature) with few researchers using a more sociocultural approach (e.g. Hodgen & Askew, 2007). I argue a sociocultural perspective on emotion could provide new insight into the phenomenon of becoming a mathematics teacher and the decision-making processes of prospective teachers. The purpose of this paper is to demonstrate the potential of a sociocultural perspective on emotion in mathematics education. I argue this by highlighting three constructs that make up an individual's emotionality (Denzin, 1984): (1) emotional labor, (2) feeling rules, and (3) emotional geographies (Hochschild, 1979, 1983; Hargreaves, 2001). Examples of these constructs in mathematics teacher education are provided from a study on the emotional experiences of prospective elementary teachers (Gomez, 2016). First, I emphasize the differences between a cognitive perspective and a sociocultural perspective on emotion.

Cognitive and Sociocultural Perspectives of Emotion

A cognitive perspective focuses on the individual's emotion while doing and learning mathematics with minimal considerations of the social other. Emotion is only considered to be "a feeling or state of consciousness" (Philipp, 2007, p. 259). Using a cognitive perspective centers around the feelings expressed by the student while doing mathematics (e.g. Goldin, 2014; Martinez-Sierra & Garcia-González, 2016). The goal of the researcher then is to understand and theorize the emotional experience and how the emotions arise after a stimulus, usually a mathematics problem. For example, Goldin (2014) described *affective pathways* as being sequences emotions occur in when doing a task and was part of the individual's affective architecture. He used affective pathways to explore the complicated patterns of emotions teachers and students exhibit during mathematical problem solving. Goldin (2014) argued researchers need to focus on "more complex descriptions of affective architecture" (p. 405) in mathematics education. The cognitive perspective is useful but limited because it is intended to only describe what elicits emotions and not how the emotional experiences influence the interactions with the social other. If we consider learning to be based on participation (Lave & Wegner, 1991) then the cognitive perspective on emotion is limited in its impact.

Mathematics education research needs to go beyond the emotional reactions of students and teachers to understand how emotion influences the learning and teaching of mathematics. A sociocultural perspective involves the exploration of how emotions influence one's interaction with agents, objects, or events and their work spaces (Denzin, 1984; Hochschild, 1983). Emotions occur in social spaces and emotional reactions are directed at something (Cross & Hong, 2012). More importantly, where these emotions are directed reflect the positioning of the individual in relation to a phenomenon (Hochschild, 1983). This necessitates defining emotions as more than just "states of consciousness" (Philipp, 2007, p. 259). Denzin (1984) considered emotion to be a major part of the interaction between individuals:

> Emotion is a lived, believed-in, situated temporally embodied experience that radiates through a person's stream of consciousness, is felt in and runs through his body, and, in the process of being lived, plunges the person and his associates into a wholly new and

> transformed reality—the reality of a world that is being constituted by the emotional experience. (Denzin, 1984, p. 66)

Recognizing emotions are used as communicative tools, they change the ways individuals interact with one another and transform the phenomenological experience of both the one emoting and the social other interpreting the emotion.

Mathematics education researchers have not taken up the sociocultural perspective of emotion explicitly, though previous studies have addressed emotions as part of a broader focus on students' interactions and mathematics learning. For example, Cobb, Yackel, and Wood (1989), related the emotional acts of children to the norms constructed in the classroom. Cobb et al. (1989) became conscious of emotional acts as "essential features of the dynamic, self-organizing social system that characterized life in the classroom" (p. 117). This revelation led Cobb and colleagues to further investigate these particular emotional acts when students were engaged in problem solving. Cobb et al. (1989) argued students' emotions in the classroom "are not only warranted in specific situations but, at times, ought to occur" (p. 119). The social norms of the classroom played a vital role in the exploration of the emotional acts of the students.

Methodology

The sociocultural perspective on emotion guided a qualitative study into the emotional experiences of prospective elementary teachers learning to teach mathematics. The goal of the study was to investigate the identity development as mathematics teachers of four prospective elementary teachers through their narratives of emotional experiences in their practicums. The four participants (Anastasia, Elsa, Sally, and Kida) were interviewed three times at the beginning, middle, and end of their first practicum experience. They also meet four times as a small group to write and discuss their experiences. All interviews and small group meetings were video and audio recorded and transcribed. Narratives were coded based on the emotional geography (see below) enacted. For further details on methods and analysis see Gomez (2016). For this report, I focus on the experiences of Anastasia. Anastasia was a white 20-year-old female in her third year at a large research University. For her practicum experience, Anastasia was assigned to work in Ms. Blaileen's 5th grade class once a week at Amos Moses Elementary.

Emotional Labor, Feeling Rules, and Emotional Geographies

Denzin (1984) argued research on emotional experiences need to focus on emotionality or the processes of emotion. Emotionality places an emphasis on how emotions influence social interactions and decision-making.

> Emotionality arises out of inhibited, interpreted social acts in which the subject inserts self-conversations between the perception of experience and the organization of action. In these conversations, feelings directed to the self mediate action and interpretation. Emotionality becomes a social act lodged in the social situation. (Denzin, 1984, p. 224)

Investigating emotionality can help in understanding the ways emotions are influencing an individual's decision-making processes and is the basis of the sociocultural perspective of emotion because emotions occur in social spaces (Cross & Hong, 2012). In this section, I describe and provide examples of three constructs used to explore emotionality: Emotional labor, feeling rules, and emotional geographies.

Emotional Labor

Individuals control his or her expressed emotions to be portrayed by the social other as a particular person (Hochschild, 1983). This identity management requires emotional labor, or "the act of trying to change in degree or quality an emotion or feeling" (Hochschild, 1979, p. 561) The institution one desires to work in can aid in developing the individual's beliefs about how to regulate one's emotions. Hochschild (1983) gave the example of airline stewardess in training:

> The pilot spoke of the smile as the *flight attendant's* asset. But as novices… move through training, the value of a personal smile is groomed to reflect the company's disposition… Trainers take it as their job to attach to the trainee's smile an attitude, a viewpoint, a rhythm of feeling that is, as they often say, 'professional.' (Hochschild, 1983, p. 4, emphasis in original)

The stewardesses saw this emotional regulation as necessary in order to be professionals. In education, Williams-Johnson et al. (2008) claimed teachers plan out, consciously or unconsciously, the emotions they will portray to students to better their own student-teacher relationships. Teachers, like other professions who serve public interests, participate in emotional labor and must learn to manage their emotions in the classroom (Schultz & Zembylas, 2009).

During Anastasia's time at Amos Moses, she was working with students who lived in a low SES area. She conducted her emotional labor when confronted by students living in poverty. She felt powerless and described the experience as "heart breaking." Regardless, she felt it necessary to conceal how she really felt:

> You just have to like paint a smile on and pretend like you are okay with everything in your classroom… Some of the things that these kids go through, I literally cannot imagine. Like one kid comes to school in the same pair of pants every single day. And like, you can't—you can't grimace and be grossed out by it. You have to be okay with it. (Anastasia, Small Group Meeting 4)

Anastasia wanted to create a positive learning environment where students' mathematics is challenged. Her emotional labor was necessary in constructing the learning environment she believed would be best for students to learn mathematics conceptually and develop the relationship she wanted students to have with mathematics.

Feeling Rules

Hochschild (1979) described feeling rules as "the social guidelines that direct how we want to try to feel may be describable as a set of socially shared, albeit often latent (not thought about unless probed at), rules" (p. 563). These rules are socially constructed from the individual's interpretation of collective understandings of appropriateness or interpretations of what it means to act professional. Emotional labor is involved in following feeling rules (Hochschild, 1983). Teacher education programs, like the airline stewardess training, influence prospective teachers in the construction of feeling rules one needs to follow to be seen as a professional.

Anastasia discussed explicitly three feeling rules she followed while at Amos Moses: (1) showing emotion will remove authority, (2) someone else's classroom is not the place to show particular emotions, and (3) modification and construction of feeling rules will continue as career begins. Anastasia's feeling rules were a consequence of becoming more aware of the complexity of being a mathematics teacher. By the third interview, when she asked directly about her emotional labor, Anastasia stated her ideas of professional behavior, including the feeling rules followed by teachers. She believed teachers lose their authority when they show emotions. "And when I think of a figure of authority I do not think of someone who's, you know, emotional." (Anastasia, Int. 3). Anastasia saw her feeling rule as positioning her as an authority figure.

Anastasia's other feeling rules were constructed as a consequence of the power struggle between her and Ms. Blaileen. Anastasia was determined to teach mathematics conceptually, but felt unable to because her desired way of teaching mathematics was not supported by Ms. Blaileen. Anastasia claimed it was not her place to push back and challenge the norms in Ms. Blaileen's classroom. This included the emotions Anastasia felt while working with students.

> I tried to stay positive because I just did not feel that was the place for me to be frustrated or discouraged or embarrassed or any of those emotions. I thought that in someone else's classroom I kind of feel like you just have to suck it up and grin and bear, wait till you get home. You know what I mean? And I'm not really sure what the like protocol for that is. It's not really something we talk about in our classes. (Anastasia, Int. 3)

Anastasia believed because she was not the authority in the classroom she could not express particular emotions. Unsure of the protocol, Anastasia leaned on her teacher education program. This influenced Anastasia to concede her construction of feeling rules to when she became the authority and could learn from a community of professionals.

Emotional Geographies

Hargreaves (2001) developed the emotional geographies framework from an analysis of 53 in-service teachers' narratives describing the highs and lows about their relationships with students, parents, and other colleagues. The emotional geographies are the spaces in which bonds between people, objects, or events are created or broken depending on the similarities or differences of a purpose (moral geography), the work of the individual (professional geography), the power position (political geography), a social aspect (sociocultural geography), or proximity (physical geography) (Hargreaves, 2001). The five emotional geographies emerged from the narratives of teachers; however, the participants ranged in grade level (elementary, middle, and secondary) and content focus. In this study, all the participants were focused on the same grade level and the interviews were focused on the teaching of the same content (mathematics). Further refinement of the emotional geographies work is needed. For this report, I describe the characteristics of Anastasia's emotional geographies based on the use of the general framework.

Moral Geography. Anastasia's moral geography was characterized by her desire to teach conceptually. This was evident from the amount of frustration, anger, helplessness, etc. she felt when observing her mentor teacher teach and the pride and validation she experienced when seeing students' success in learning conceptually. Anastasia was particularly proud of her experience working with one student stating: "I think that seeing him succeed…and then the next day or the next week still being able to do that and build upon that. I mean that's what changed—made it such a priority to me. To do the conceptual with kids" (Anastasia, Int. 3).

Professional Geography. Anastasia's professional geography was characterized by: (a) Curriculum decision-making processes and (b) Being able to communicate the mathematics to students. Anastasia saw both of these aspects as being influential to her capacity to teach mathematics conceptually. She felt both aspects of her professional geography needed further development, and she saw her coursework as supportive of her desired trajectory.

Political Geography. Anastasia's political geography was characterized by the power struggle between Anastasia and her mentor teacher and the influence on Anastasia's decision-making processes. Many times, Anastasia made statements like, "at the end of the day it is her classroom. So I can only do what I think is best for the kids within her parameters." (Anastasia, Int. 3).

Sociocultural Geography. Anastasia's sociocultural geography was limited to her awareness of gender issues. Reflecting on her experiences in elementary and middle grade mathematics, Anastasia referred to being selected for her school's mathematics team and being the only female on the team. The social standing achieved by being good at mathematics at her school may also have validated her feelings of pride. This awareness only emerged when reflecting back on her own education, and not when discussing her experience at Amos Moses Elementary.

Physical Geography. Anastasia's physical geography was characterized by her limited presence in the classroom. Anastasia felt her limited amount of time in the classroom was problematic for building a relationship with Ms. Blaileen. "I understand it would be hard to trust people that come into your classroom once a week" (Anastasia, Int. 2).

Anastasia's emotional geographies characteristics demonstrate the ways her emotionality influenced her ways of thinking about teaching and learning mathematics. The characteristics also influenced what Anastasia saw as possible for herself as a teacher. These characteristics constructed the boundaries and constraints Anastasia had to work within to form her conceptualizations as a teacher-of-mathematics.

Discussion & Conclusion: The Benefits of a Sociocultural Perspective of Emotion

The sociocultural perspective of emotion provides a different way to consider the influence of emotions on becoming a mathematics teacher. The feeling rules Anastasia constructed were not specific to mathematics nor were the other participants' in the larger study (Gomez, 2016). The experiences in the mathematics classroom, although, did influence the construction of the general feeling rules. Furthermore, the characteristics to her moral geography were the only ones specific to mathematics. I conjecture that due the participants being early in their course work, most of the reflection work conducted was guided to developing their purpose as mathematics teachers. As they continue to visit schools and prepare for student teaching, the moral geography will have greater influence. We can see from Anastasia's case that the moral geography influenced the professional and political geographies because it defined Anastasia's purpose for teaching mathematics, but the characteristics were still yet specific to mathematics.

Future work can focus on specific constructs of the sociocultural perspective of emotions. Considering feeling rules, one can determine the possibility of the construction of socioemotional mathematical norms or the norms set in a classroom about how students should feel about doing and learning mathematics. Also, emotionality can be explored with a different theoretical lens, like a situative perspective (Lave & Wenger, 1991) to refine and expand the constructs of emotional labor, feeling rules, and emotional geographies specifically to mathematics education.

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FOLLOWING THE TRAIL OF MATHEMATICS ANXIETY FROM PRESERVICE TO IN-SERVICE

Gina Gresham
University of Central Florida
Gina.gresham@usf.edu

This study provides an in-depth look at in-service teachers' mathematics anxiety, the effects teaching experience may have as to its causes, the strategies used to address alleviate and/or overcome it after 5 years teaching experience. Interviews, and a 24-item self-rating Mathematics Anxiety Rating Scale-Revised (MARS-R) indicate that over 95% had either moderate to high levels of mathematics anxiety ($p < .001$). This research provides support to the importance of mathematics methods courses, collaborative experiences and teaching practices, and a glimpse into in-service teachers' personal impact on their students.

Research on mathematics anxiety in both preservice and in-service elementary teachers (Adeyemi, 2015; Aslan, 2013) has focused on the origin of prior negative mathematical experiences, mathematical beliefs, the effect of prior teachers, and teacher education training programs. An underlying assumption of this research is that preservice teachers with high levels of mathematics anxiety are likely to become teachers who do not enjoy teaching mathematics (Gresham, 2009). Thus, they continue avoidance of the subject due to lack of confidence, ability, and mathematical content knowledge (Beilock & Maloney, 2015). This longitudinal research study revisited in-service teachers (previously involved in a preservice teacher research study on mathematics anxiety) after 5 years teaching experience. The purpose was to (1) determine and compare in-service elementary teachers' levels of mathematics anxiety to their preservice posttest mathematics anxiety scores after 5 years teaching experience, (2) determine if participation and preparation in a preservice mathematic methods course continued to affect in-service elementary teacher's mathematics anxiety, and (3) to determine, if any, the cause of such affected change in mathematics anxiety.

Mathematics Anxiety, Preservice & In-service Teachers, and Instruction

Mathematics anxiety knows no boundaries (Tobias, 1978) and has been defined as a feeling of uncertainty and an uneasiness when asked to do mathematics and the "I can't" syndrome. It is an inability to perform well on tests, a feeling of physical illness, helplessness and panic, faintness, and mental disorganization (Bursal & Paznokas, 2006). It is a phenomenon where individuals suffer from the irrational fear of mathematics to the extent they become paralyzed in their thinking and are unable to learn or be comfortable with mathematics.

Extensive research on mathematics anxiety has tried to determine why so many people (particularly females) in the United States demonstrate a fear or even antipathy toward mathematics (Belbase, 2013; Lake & Kelley, 2014). A study by Aslan (2013) indicated that in-service teachers had higher levels of mathematics anxiety. Lake and Kelly (2014) linked negative teacher attitudes about mathematics to mathematics anxiety. Negative attitudes toward mathematics and mathematics anxiety can produce negative results in mathematics due to the reduction of effort expended toward the activity, the limited persistence one exerts when presented with an unsolved problem, the low independence levels one is willing to endure, and whether or not a certain kind of activity will even be attempted (Vinson et. al, 1998; Fetterly, 2010). Obviously, limited teaching experiences, coupled with high levels of mathematics anxiety in preservice teachers is a concern as it may certainly carry over into the classroom once they become in-service. As a result, it is believed that teachers must be adequately prepared in mathematics and should be done through mathematics methods courses offered by teacher education programs (Aslan, 2013; Lake & Kelly, 2014).

Participants

Ten elementary in-service teachers participated in a prior study (involving 267 preservice teachers) that investigated preservice teachers' mathematics anxiety before and after a mathematics methods course. These ten teachers maintained the highest mathematics anxiety rating in the preservice study and were therefore, chosen for the in-service study. As undergraduates, all were required to take college algebra, principles in statistics, one K-6 mathematics content course, and K-6 mathematics methods course. No teachers involved in this study chose to enroll in courses beyond what was required within the undergraduate elementary education program. Three teachers held a master's degree in education and one was within weeks of graduating with a master's degree in education. All were female and taught in self-contained public schools. Seven were Caucasian, 2 were African American, and 1 was Hispanic.

Data Analysis

Paired sample t-tests were completed to consider differences between the pre- and posttest Mathematics Anxiety Rating Scale (MARS) levels from the preservice scores (prior study) and from the posttest- in-service test scores (current study). Scores from the preservice study were subtracted from the in-service MARS score to reveal a difference in scores. A positive difference score meant that the in-service teacher's mathematics anxiety actually increased after 5 years teaching experience. A negative score meant that the in-service teacher's mathematics anxiety continued to decrease by that much.

Informal observations of teachers, questionnaire-guided narrative interviews, informal discussions and interviews were initiated by one researcher (professor) in both studies. The interviews were usually in response to questions regarding their own personal concerns, experiences, background, assignments, and mathematical teaching practices. Field notes and audio recordings of interviews and discussions were used and analyzed and decoded for emerging themes.

Quantitative Findings

Table 1 illustrates preservice teachers' mathematics anxiety scores from the previous study. The MARS pretest/posttest comparison scores and gains revealed significant decreases in all 10 teachers' mathematics anxiety from pre-posttest after participating in a mathematics methods course. Table 2 shows the comparison of the preservice posttest and in-service test raw mean scores from the in-service teacher study. After comparing group means for the posttest and in-service scores, it was found that overall in-service teachers' mathematics anxiety (although still highly prevalent) decreased slightly after 5 years teaching experience. Teachers 1 and 2 had the least mathematics anxiety reduction after 5 years teaching experience with a -09 and -08 decrease respectively thus maintaining a high anxiety grouping in both studies. Teacher 5 had the continued greatest reduction of mathematics anxiety with a -41 gain in scores. Teachers 3, 4, 9, and 10 had decreasing scores ranging between -13 to -16. Teacher 6 and 8 had a -27 decrease Teacher 7 had a -26 decrease in mathematics anxiety scores.

Qualitative Findings

All 10 teachers commented that their mathematics anxiety was consistently evident in their mathematics classroom throughout their 5 years teaching experience. Each identified daily struggles not only within themselves but in their students as well. Comments revealed their mathematics anxiety and lack of mathematical confidence and mathematics skills affected their classroom practices. Several teachers expressed a disdain towards mathematics as "very overwhelming to the point of madness." First and second grade teachers revealed they felt more comfortable teaching the lower grades because they lacked confidence with the mathematical content to teach in the upper grades.

Table1

MARS Preservice/Posttest Comparison Scores

Teacher	Grade Level	Preservice Pretest Score	Preservice Posttest Score	Gain
Teacher 1	1st	331	260	-71
Teacher 2	2nd	343	269	-74
Teacher 3	2nd	273	212	-61
Teacher 4	2nd	297	218	-79
Teacher 5	3rd	328	250	-78
Teacher 6	4th	299	231	-82
Teacher 7	4th	296	228	-68
Teacher 8	4th	300	216	-84
Teacher 9	5th	275	202	-73
Teacher 10	5th	283	205	-78

Note. $p < .005$

Table 2

MARS In-service Comparison Score

Teacher	Grade Level	Preservice Posttest Score	In-service Test Score	Gain
Teacher 1	1st	260	250	-09
Teacher 2	2nd	269	261	-08
Teacher 3	2nd	244	229	-15
Teacher 4	2nd	238	223	-15
Teacher 5	3rd	250	209	-41
Teacher 6	4th	231	204	-27
Teacher 7	4th	228	202	-26
Teacher 8	4th	216	199	-27
Teacher 9	5th	202	186	-16
Teacher 10	5th	205	192	-13

Note. $p < .001$

Both teachers indicated such a strong "dislike and fear towards mathematics" that implementing effective lessons in the upper grades would be "intimidating and ineffective which would certainly negatively affect their students." Teachers 6, 7, and 8 (each earning a master's degree in education by their 4th in-service year) also had significant decreases in scores after 5 years teaching experience. All three reiterated that additional mathematic courses (taken with the same professor as their undergraduate) and the required professional development mathematics workshops in their graduate program of study highly impacted their attitudes and confidence within the mathematics classroom and improved their overall content knowledge. However, Teachers 5, 6, 7, & 8 (all 4 master's teachers) expressed the need for "career long" mathematics professional development to continue strengthening their mathematics skills and effective mathematics teaching practices while specifically addressing and/or alleviating their mathematics anxiety in the process. Even though all 10 teachers' mathematics anxiety was still highly evident all involved felt the teacher's attitude toward the subject set the tone for students' successful learning. Each felt their mathematics anxiety decreased as they gained more teaching experience as they worked to *hide* their negative feelings regarding mathematics and employ a positive attitude while in the classroom. Teachers saw "themselves in their students" with relation to their own mathematics anxiety. They also emphasized how much they learned during the methods course and viewed the activities, strategies, etc. that were incorporated within the methods course as ones currently employed within their mathematics classrooms and which were beneficial in helping reduce their mathematics anxiety. They proved diligence in implementing lessons that were engaging and hands-on with real world applications to spare their students the same angst they experienced while learning mathematics.

In the prior study, 4 themes emerged during preservice interviews that related to teachers' perceptions towards the effectiveness in teaching mathematics to elementary students. These themes included attitudes towards mathematics, mathematics teaching practices, description, and understanding of mathematics. In-service teachers were asked to read their preservice teacher comments to determine if their prior perceptions regarding mathematics anxiety had changed. Their current comments were used to compare their thoughts after 5 years teaching experience. From the prior study, preservice teachers' negative attitudes towards mathematics surfaced very quickly. Five years later all 10 teachers indicated they still did not like mathematics and really struggled with the subject and their negative attitudes. Many in-service teachers' statements described negative emotions with words such as "stressed", "embarrassed", "frustrated", "fearful", "discouraged", and "struggling." They associated these words directly with their personal mathematical experiences not only as a former student themselves, but as a teacher. Their words suggest confidence inadequacies in their ability to effectively reach their students both academically and emotionally and identifies their well-entrenched beliefs about mathematics teaching and learning. In-service teacher recognized their fears for a lack of solid understanding of mathematical content. They expressed a much needed intentional change once they saw how their attitudes and frustrations affected their students. They reflected on the outcomes as having an effect which required a change in their teaching practices through the implementation of the methods learned in the preservice experience. The existing classroom practices were explained as was the development of somewhat more realistic expectations for successful learning outcomes. Their comments are evident of their attempts to self-motivate and confidence build while forging ahead with efforts to establish a more positive learning environment in spite of their mathematics anxiety, struggles, and frustration. All 10 teachers, regardless of their level of mathematics anxiety, reiterated the importance of using manipulatives, engaging, and motivating students for successful mathematical learning. They also felt that using manipulatives contributed to lessening their own mathematics anxiety to some degree. Although a few in-service teachers stated they were unsure of their use as a preservice teacher and at the very beginning of their teaching careers, they felt that as time passed they became more comfortable with teaching and with incorporating manipulatives as an essential learning tool in the classroom. As a preservice teacher, some described mathematics and their understanding of it in different ways. Although many attributed it to their ability to work hard, memorize material, and from positive experiences at home and school, others commented on the struggle to learn mathematics, lack of help from parents who had little knowledge of the mathematical content, and the challenge to meet their weaknesses in mathematics. As an in-service teacher, they not only reiterated the same thoughts but that “drill and kill” of mathematics content did not lead to mathematical understanding.

In-service teacher interviews showed emerging themes including: the significance in taking the undergraduate mathematics methods course, how the methods course affected their instructional practices, how the mathematical practices learned in the course are the practices currently used in their classrooms, and the demands of teaching. Teachers acknowledged need for deep understanding of mathematics for their effectiveness in making meaningful contexts which are constructed, connected, and applied to mathematical learning. They felt students must be actively engaged and students must be participating in the learning and understanding of mathematics. Their comments show that such use would tend to promote problem-solving ability by providing a vehicle through which children can model real-world situations as illustrated in the methods courses. They also stressed the importance of how manipulative materials can be effectively used as an intermediary between the real world and the mathematical world.

Discussion and Conclusions

Even though the decrease was minimal, the quantitative results from the study concludes that teachers' math anxiety was reduced through their teaching practice. The intention of this study was to provide a more in-depth look at in-service teachers' mathematics anxiety and the effects teaching experience may have with regards to their mathematics anxiety. From this study, we are reminded through the teachers' voice of how teacher mastery of the mathematical content also has an effect on the students (Beilock, & Maloney, 2015) as it goes back to teacher preparation and knowledge of subject matter. Teacher comments indicated that they longed to situate the course and their experiences within their development as an elementary educator and for direct applications to the elementary mathematics classroom. Therefore, the results from the study provides insight into the durability and effectiveness of teacher training programs that emphasize manipulatives and other strategies to help reduce mathematics anxiety in teachers (Aslan, 2013). This research provides support to the importance of mathematics methods courses, collaborative experiences, and the use of manipulatives in shaping preservice teacher's eventual practices as an in-service teacher as well as a glimpse into in-service teachers' recognition of their personal impact on their students. Even though in-service teachers' mathematics anxiety did continue to slightly decrease after 5 years teaching experience, the greatest decrease in mathematics anxiety occurred while involved in the mathematics methods course as a preservice teacher. The usefulness of the mathematics methods course and experiences was a salient finding. Therefore, this study further supports the importance of having teacher education programs that influence the development of effective instructional practices while specifically addressing the reduction of mathematics anxiety in preservice teachers.

Even though this study is situated in the context of a smaller sample size, the argument has demonstrated the complexity of mathematics anxiety as a universal concern for all mathematics educators. When considering the findings, a determination is not made that changes will persist or will continue to change. However, carefully examining the process of change even with this study's sample size may help us become better informed not only about the longitudinal effectiveness of our mathematics methods course but the usefulness in understanding the important outcomes of those mathematics methods courses across time. The study results also provide a foundation for more investigation of the need for continued mathematics professional development opportunities that specifically address mathematics anxiety and to determine how in-service teachers' mathematics anxiety influences and affects instructional practices. This study will continue with the same in-service teachers at 10 and 15 years and/or longer to further research and discover how and/or why in-service teachers' cycle of mathematics anxiety continues or changes and to identify all causes of such change as service years increase. Studying the longitudinal effects of mathematics anxiety in preservice teachers is a critical component for institutions of higher education to make informed decisions about the mathematics methods courses included in teacher education programs. Researchers and teachers must continue to work together to determine which curricula and instructional practices will bring the best results in mathematics achievement and thus eliminate the cycle of mathematics anxiety.

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BARRIERS TO IMPLEMENTATION OF CONSTRUCTIVIST TEACHING IN A HIGH-POVERTY URBAN SCHOOL DISTRICT

Danya M. Corkin
Rice University
dmc7@rice.edu

Adem Ekmekci
Rice University
ekmekci@rice.edu

Stephanie L. Coleman
University of Houston-Downtown
colemanst@uhd.edu

Guided by the "constructivist in practice" dilemmas framework developed by Windschitl (2002), we investigated the conceptual, pedagogical, cultural, and political barriers that K-12 mathematics teachers in a high-poverty urban district encounter when trying to implement constructivist practices they learned through a rigorous professional development (PD) program. Themes that emerged from this qualitative analysis included barriers concerning lack of awareness of constructivist theory, difficulties facilitating constructivist activities in the classroom, poverty, lack of instructional autonomy, and high-stakes testing. Identifying barriers to constructivist teaching may inform teacher educators and school administrators in developing strategies to overcome these obstacles and improve mathematics-teaching quality.

Introduction

At its core, constructivist teaching facilitates an active learning environment where students interact with one another and connect new ideas with existing knowledge to construct a meaningful conceptual understanding of information within an academic discipline (Hennessey, Higley, & Chesnut, 2012). Within the mathematics education community, the constructivist teaching philosophy serves as the framework for reform-based teaching (National Council of Teachers of Mathematics, 2014). Research indicates that this philosophy is associated with greater student achievement in mathematics, enhanced algebraic procedural and conceptual understanding, as well as more sophisticated epistemological conceptions of mathematics (Kim, 2005; Ross & Willson, 2012; Star & Hoffman, 2005). Thus, high-quality teacher professional development aims to develop the conceptual and pedagogical groundwork for mathematics teachers to implement constructivist practices in their classrooms (Garet, Porter, Desimone, Birman, & Yoon, 2001). However, even when mathematics teachers gain the adequate conceptual and pedagogical foundation to implement constructivist practices, they may still face political and cultural challenges to enact these practices in their classrooms (Windschitl, 2002). These challenges may be more pronounced in high-poverty urban schools where emphasis on rote learning, scripted lessons, mandated curriculum, and accountability is more likely to prevail (e.g., Crocco & Costigan, 2007). Perhaps because of these challenges, research has found that teachers working in high-poverty urban school schools are less likely to enact constructivist instructional approaches that align with mathematics reform standards compared to their counterparts teaching in low-poverty school districts (Berry, Bol, & McKinney, 2009).

Therefore, guided by the “constructivist in practice” dilemmas framework developed by Windschitl (2002), we will investigate the personal, cultural, and political barriers that K-12 mathematics teachers in a high-poverty urban district encounter when trying to implement constructivist practices taught through a teacher professional development program. In addition, we seek to identify motivational and behavioral strategies teachers utilize to overcome these obstacles to sustain constructivist practices. It is the hope that through this research, we will help further the quality of mathematics instruction, and, in turn, students’ mathematics achievement.

Theoretical Framework

To address the dearth of research that seeks to uncover the full spectrum of challenges teachers face in facilitating constructivist classroom environments, Windschitl (2002) developed the “constructivist in practice” dilemmas model to propose that there are four broad dimensions that capture challenges to the implementation of constructivist teaching: conceptual, pedagogical, cultural, and political dilemmas. Conceptual dilemmas involve teachers’ epistemological understanding of constructivism. Pedagogical dilemmas deal with the design of curriculum and classroom activities to align with constructivist teaching. Cultural dilemmas involve the roles that are necessary among teachers to facilitate a constructivist classroom environment. Political dilemmas are encountered when resistance to constructivist teaching arises among various stakeholders within school communities (Windschitl, 2002). This framework will serve as an initial guide to organize our qualitative interview data concerning teachers’ barriers to implementing constructivist practices learned through rigorous PD in mathematics instruction.

Research Questions

1. What types of barriers do teachers working in high-poverty schools encounter when transferring constructivist practices learned through PD into their classrooms?
2. What types of facilitators assist teachers working in high-poverty schools to implement constructivist teaching learned through PD?

Method

Participants

A total of 80 K-12 in-service mathematics teachers from urban school districts in Texas participated in a three-week rigorous summer PD program focusing on mathematical content and pedagogical knowledge informed by constructivist theory. The teachers volunteered or were selected by school administration to participate in the program. The mathematical content focus was: (a) numbers, operations, and quantitative reasoning; and (b) patterns, relationships, and algebraic reasoning. A total of 80 teachers from eight school districts and one private school represented the initial sample. We identified 52 teachers working in high-poverty schools within a high-poverty school district. We categorized these teachers by grade level (K-6 and 7-12) and teaching experience (experienced and novice [less than 5 years of teaching]). We randomly selected two teachers from each cell of this 2X2 design (8 teachers in total).

Procedure

Authors developed a structured interview protocol that included questions about their experience in the PD program, their teaching philosophy, and barriers and/or facilitators to implementation of teaching practices of what they have learned through the PD. Authors interviewed these eight teachers in the spring semester of the 2015-16 academic year following the summer PD. Student research assistants transcribed the interviews. All three authors read through the interviews to identify specific manifestations of Windschitl's (2002) four constructivist dilemmas in the transcripts and met to discuss the themes developed (Patton, 2002). Then, authors categorized these themes within Windschitl's (2002) four dilemmas of constructivist teaching. In addition, within each type of dilemma authors specifically made note of factors that either helped (facilitator) or hindered (barrier) their use of constructivist teaching methods. Authors used Windschitl's (2002) descriptions broadly and included examples that were not explicitly mentioned in the article but were consistent with the overall conception of the dilemma. After developing a first draft of a codebook that included detailed description of the codes, we selected two interviews at random to be coded by all three authors. A second coding meeting was held to discuss what codes authors had applied and why. If there was a discrepancy, authors resolved them to establish interrater reliability. Additional revisions of the codebook were made based on authors understanding of the codes as they were applied to the interview. After establishing agreement and finalizing the codes, each author coded four interview transcripts so that each transcript was coded by two authors. A final meeting took place between the pairs of authors to resolve any coding discrepancies.

Findings

Below we describe Windschitl's (2002) four dilemmas and how each one manifests for participating mathematics teachers in high-poverty urban school districts.

Conceptual barriers. According to Windschitl (2002), conceptual dilemmas refer to the difficulties in understanding the constructivist approach to teaching. Teachers' deep understanding of the constructivist approach might be thought of as a predecessor to effective constructivist techniques because of the inherent philosophical nature of the approach as well as the significant departure from "traditional" teaching methods. Furthermore, teachers may conflate the activities associated with constructivism with the approach itself. In other words, they may implement ostensibly constructivist methods (use of manipulatives, social dialogue) without implementing the core of constructivist theory because of poor understanding of the theory itself.

In our data, we looked for evidence of conceptual barriers, such as teachers being confused about or unaware of the term "constructivism" when asked directly about their opinions about constructivism. Conceptual barriers also included teachers believing that students learned math best via traditional methods. For instance, one teacher noted, "I am old school in that I do like them to learn paper/pencil first before we move on to the calculator."

Conceptual facilitators. Conceptual facilitators included endorsing beliefs and implementing instruction that are consistent with a constructivist approach, such as developing understanding through social interaction and implementing student-centered approaches. Though teachers were often unfamiliar with the term "constructivism" and were not able to fully articulate the theory, many teachers conveyed that students learn math best through methods and activities consistent with constructivism. For instance, one teacher said,

> *So, I like my kids to learn through play because I think that that works best for kids. Kids learn through each other and they learn through play and they learn through conversation (…) Numbers are just symbols, but if they don't understand what it means. They are not able to manipulate it, then they are not able to do math, because if they don't understand it, then it is just essentially rote versus actually knowing it, and actually being able to argue why you are doing it and then me teaching them a strategy and then they coming up with their own strategy, versus me teaching them this strategy and they are using that without going deeper. So, I like to teach them to think deeper and dig deeper through hand-on interactions and conversations with each other.*

Pedagogical barriers. Windschitl (2002) describes pedagogical dilemmas as teachers' application of constructivist theory to the learning environment via tasks and activities. Specifically, these dilemmas refer to teachers' attempts at transforming their instructional technique from traditional, didactic methods to methods consistent with constructivism. As part of this process, teachers must shift their focus from, for example, supplying answers and techniques to acting as a facilitator for student learning; from minimal student interaction to facilitating academically productive student dialogue; or from using pre-determined problem sets to creating complex problems that provoke deep and meaningful work. Pedagogical dilemmas can also refer to teacher attributes, such as deep background knowledge or interest in the material.

In terms of pedagogical barriers, some teachers mentioned the difficulties in facilitating activities, such as the use of manipulatives, noting that students can use them inappropriately. Other pedagogical barriers included teacher attributes, such as poor motivation ("it's about maintaining my motivation, cuz this is a burn-out industry, and I felt it. You know, I have felt it.") or lack of background knowledge ("And I'm not like a mathematician, like some people are. They get these concepts and they're real fast, and they get it, and their knowledge is real deep right away, and not so with me.").

Pedagogical facilitators. In terms of pedagogical facilitators, several teachers noted that they were able to successfully manage the different demands of a more constructivist classroom (several mentioned coping with the "organized chaos" involved in having students work more cooperatively). There were several different types of activities mentioned, such as using manipulatives, technology, art, real-world data collection, and even yoga to teach math. One teacher describes her use of manipulatives as follows:

> *As an 8th grade teacher I always think that my kids are too old for manipulatives. And, there were some really good activities that we did over the summer that I don't know. And I did them in my classroom. And I don't know how I would've done them without the manipulatives. Um, you're never too old. As students, you're never too old for manipulatives, it's just the process changes. And, you know, I used the algebra tiles with my algebra kids. I've used the two color counters with my 8th grade kids. We did a thing with my 8th grade kids, that one of my instructors did with bags for the real number system. And my kids, they got it. Because they had that visual there to see. So, it was really fun. I enjoyed it.*

Cultural barriers. Windschitl (2002) defines cultural dilemmas as occurring when teachers encounter difficulties related to learning expectations from students or other stakeholders that do not match constructivist theory. For instance, part of implementing constructivism entails re-writing the “unwritten rules” related to participation and decision-making. The cultural background and expectations of students and even fellow teachers may also be inconsistent with a constructivist approach, creating difficulties in implementing the approach.

In our data, cultural barriers included other teachers’ attitudes as favoring status quo teaching approaches, difficulty implementing the constructivist techniques learned through the PD due to classroom management concerns, or student poverty as a barrier. For instance, one teacher noted, “So the distractions, just just the fact that outside of these walls, there's nothing to motivate them to do what we're doing here. You know, it's just, they're just, they're on survival mode out there.” Related to the attitude of other teachers, one teacher noted:

> *Some older teacher, I guess, veteran teachers are gung-ho on having their kids memorize these facts, and I understand memorizing the facts once you know what they mean. But they are like no, I am going to drill and kill. 9 x 7 is what? 5x5 is what? And the kids don't understand the concept and it’s mainly because I guess the teachers that they had before didn't do what they needed to do to develop the concept (…) So, I am not for drill and kills, some people are. I understand why you shouldn’t do them. I understand why you should. But some people are like adamant. They are like nope, I am gonna do drill and kill it’s always worked and I am gonna continue to do it.*

However, other teachers noted their colleagues’ attitudes and behaviors as facilitating their instruction. For example, one teacher said:

> *The other teachers in my immediate area, just around me, we're really amazing support system for each other, we keep an eye out for each other, we know the ins and outs of what's going on and what our deeds are, what students need help with in what periods, it's really about building that team around you.*

Political barriers. Windschitl (2002) discusses political dilemmas as occurring when systemic barriers interact with the implementation of the constructivist approach. These interactions can occur with a variety of stakeholders, such as campus or district-level administrators or from parents or other community stakeholders. Transitioning from the “traditional” and somewhat expected framework of instruction is apt to produce controversy and tensions amongst a variety of stakeholders. Teachers did report some political facilitators, such as some degree of administrative support (“my principal just kinda lets us go teach, do what you need to do.”). However, there were a number of political barriers that were mentioned by the teachers interviewed.

One of the most common political barriers came when teachers were not able to access the instructional resources they needed in order to implement constructivist learning in their classrooms. In implementing the things they learned through the PD, teachers would need access to resources such as manipulatives, technology, and materials for interactive notebooks. However, teachers reported that they were unable to secure the specific resources they needed to be able to effectively implement what they learned. For instance, one teacher noted, "Well as far as math is concerned, we don't have the manipulatives in order to teach the concepts."

Another common political barrier to implementing constructivist teaching came when teachers encountered the overlapping concerns of testing, timing, and flexibility. Because of state accountability tests, these concerns are likely related given that schools and districts have come up with a specific curriculum sequence and timeline to ensure students are prepared for the tests. Schools and districts may also use prescriptive methods to ensure teachers adhere to methods they believe will result in higher test scores. These types of barriers were very commonly mentioned especially as they contrast with a constructivist teaching framework. For instance, teachers wished they had more flexibility and time to be able to explore concepts, correct misconceptions, and engage in exploratory learning but found that they encountered some pushback when attempting to deliver those types of strategies into the classroom. For instance, one teacher described a tension between what she described as "real people time" (the actual time it would take to learn a concept) and "artificial time" (the timeline dictated by the testing schedule). One teacher described some negative interactions with an administrator due to these concerns, "Like I have said, I have gotten chewed out multiple times for not being where I should be on the pacing calendar..." Other teachers noted:

> *The other thing is that sometimes we're not free to teach the way the concept was brought across in the training here. And so we basically have to adapt to whatever the campus wants to do. Like, however the campus wants to teach the concept, you know, if it's not tested on the STAAR, we don't teach it in the classroom. Or if it's one of the items that are not tested very often on the STAAR, we don't spend a lot of time on it. Even though it's going to be something that they're going to need to have a foundation in algebra for. We gloss over in 8th grade, where we really need to spend the time because it's not one of the important TEKS that will have questions.*

Discussion

This study applies and extends Windschitl's (2002) "constructivist in practice" dilemmas framework by elucidating how these dilemmas manifest for teachers working in high-poverty urban districts. Perhaps some of the most common dilemmas reported included conceptual dilemmas and political dilemmas. Results indicate that while teachers develop an understanding through PD of how students learn best that is consistent with constructivism; they still lack full awareness of their underlying teaching philosophy. Based on this finding, we recommend that teacher educators explicitly convey the theoretical framework that informs the pedagogical approaches their programs endorse. Prominent political dilemmas included lack of instructional resources and instructional time constraints due to high stakes testing. These findings imply that additional consideration by district and school administration is necessary to support teachers so that they gain maximum benefit from their constructivism-informed PD experiences.

Acknowledgement

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MATH IS NOT A RACE: ONE STUDENT'S JOURNEY FROM BASIC ALGEBRA TO DOCTORAL CLASSES

Jennifer Crooks Monastra
University of South Carolina
crooksj@mailbox.sc.edu

Sean P. Yee
University of South Carolina
yee@math.sc.edu

This case-study focuses on a cultural counter-narrative of Andrew's journey from high-school dropout to mathematics graduate student. Andrew has completed all required doctoral qualifying exams and loves teaching undergraduate students. Narrative inquiry was implemented with a semi-structured interview to determine what life experiences, and in what ways those experiences influenced his pedagogical decisions. Analyzing the interview data through four narrative cycles of coding resulted in identifying specific life experience structures that influence Andrew's teaching decisions. His experiences demonstrate strong connections to Andrew's decisions related to his presentation of mathematical content.

Introduction

To encourage more students to explore and engage in Science, Technology, Engineering and Mathematics (STEM) fields, a large amount of research has focused on students' high school preparation for STEM majors. Students enrolling in advanced science and mathematics coursework in high school have been shown to choose and complete STEM majors at a much higher rate than their peers who lack this foundation (Gottfried, 2015). However, the current STEM literature has limited narratives on students who move slowly through early mathematics courses, enrolling in introductory college algebra coursework and yet, successfully switch into a career requiring advanced mathematics coursework (Becker and Park, 2011). Our study proposes to inform the field with such a narrative.

In addition, teachers' practice is shaped by their life experiences (Bukor, 2013; Klausewitz, 2005). "To understand what happens when teacher and student meet in teaching-learning situations, it is necessary to understand their stories" (Connelly & Clandinin, 2000, p.318). Even early experiences with family have been shown to make long lasting impacts on teachers' "perceptions of their identity as well as on their instructional practice and teaching philosophy" (Bukor, 2013, p.63). Life experiences work as a filter or lens as teachers evaluate their teaching and make classroom decisions (Klausewitz, 2005). Their personal experiences can help explain why teachers make decisions in particular ways (Klausewitz, 2005) and reveal a connection between their experiences as mathematical learners and their teaching approaches (Kaminski, 2003). This case study seeks to understand how and in what ways the participant's story influences his teaching.

Our participant (Andrew), shares his journey from a high-school dropout to a mathematics doctoral candidate who loves teaching. Andrew persisted through adversity to enjoy a deep study of mathematics, passing all qualifying exams, and becoming passionate about teaching mathematics. The purpose of this study is to identify the participant's background and experiences, and understand how his life history influenced and informed his teaching of undergraduate students. We seek to answer the following research question: *What life experiences influenced, and how did those life experiences influence, Andrew's decisions about teaching undergraduate mathematics?*

Theoretical Framework

To connect Andrew's decisions and life-experiences, we focused on a portion of the conceptual framework proposed by Borko et al. (1992) for their study which sought to understand and describe how various factors influence and shape beginning middle-school mathematics teachers as they learn to teach. While Andrew is not a middle-school teacher nor was he enrolled in a teacher preparation program, several of the factors were appropriate for our research question and were significant for Andrew. Borko et al. theorized that teachers' knowledge, beliefs, classroom thinking, and actions influence one another, which is indicated by the double arrows (Figure 1).

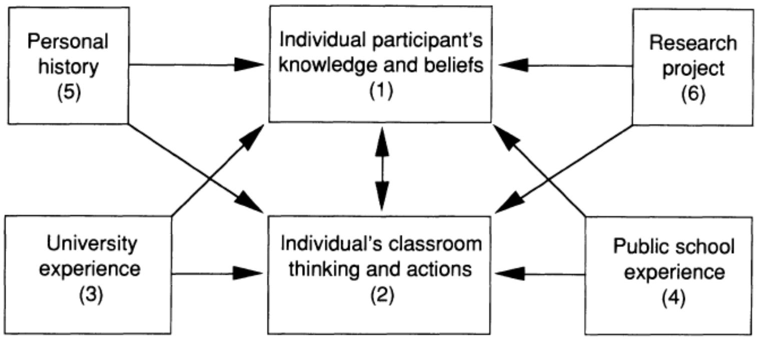

Figure 1. Borko et al.'s (1992, p. 200) influencing factors of teaching.

The teacher education program (Box 3) and their actual teaching experiences (Box 4) were considered to be significant experiences that would impact teaching. The teachers' personal histories (Box 5) and participation in a special project (Box 6) were theorized as secondary influences.

Our study seeks to understand the life and teaching experiences of a graduate student who did not experience formal teacher education and is not a certified teacher. His story was engaging because he took a non-traditional path into doctoral mathematics coursework and plans to enter the teaching profession. Thus we focused on how his life experiences (Box 5) inform both his classroom decisions (Box 2) and his feelings or perceptions related to teaching (Box 1).

Method

Narrative research aims to understand individuals' experiences over time and within context (Caine, Estefan & Clandinin, 2013). It is interested in answering questions about the meaning of experience, not just describing or giving details of an experience (Kramp, 2004). Thus, narrative inquiry was chosen because the goal of this study was to understand the participant's life experiences and how he makes meaning from them. This qualitative case study used a semi-structured interview to understand the participant's life experiences and their influences on his teaching.

Participant's Background

The participant, Andrew, volunteered for this study. Andrew's high school experiences did not prepare him for graduate work in mathematics. As a high school student, Andrew stated that he disliked school. He described himself as a student who would try to disrupt the class and who had issues sitting still. He began in trigonometry but was dropped down to business math. Ultimately, he dropped out during his senior year because he had not completed the work. "I wasn't interested in things that they were giving me in high school" and was "just bored for the most part."

When Andrew learned that he was eligible for financial aid if he attended the local community college, he quickly earned a GED and went "right away because I would get money." After making good grades on tests and papers, Andrew "found out that [he] was actually good at school." Even though he placed into an introductory algebra course, he excelled in mathematics and chose it for his major. "I literally did start, not necessarily at the bottom, but like as close to the bottom as you can get in college."

Eventually Andrew transferred to a large university and completed a bachelor's degree after eight years in higher education. Continuing onto graduate school, he even passed the qualifying exam for a PhD in mathematics on his first attempt but chose to leave with a master's degree so that he could move back to his hometown and spend more time with his wife and family.

Data Collection

An interview guide was designed to answer the research question. The first five questions focused on the participant's path into doctoral coursework. The last five questions were designed to understand how his personal biography influenced the participant's teaching. In addition, one researcher observed the participant's teaching a single lesson and the other researcher watched a recording of the same lesson. This triangulation of data allowed the researchers to deepen their understanding of the participant's teaching and confirm statements made by the participant related to his teaching behaviors.

Data Analysis

In order to answer the research question relating Andrew's life experiences to his teaching decisions, the entire interview was transcribed by the first researcher and four cycles of coding were used to understand the data and identify themes. We initially coded using Saldaña's first cycle coding methods, by primarily using descriptive coding, process coding, and in vivo codes (Saldaña, 2009). Next we paraphrased through memo coding (Willig & Stainton-Rogers, 2007), where we sought to identify transitions and themes in his story, as is appropriate for our framework of narrative inquiry. A coding scheme was designed from the first two cycles by clustering concepts. Next, we applied stanza coding (Willig & Stainton-Rogers, 2007) to verify that our codes for experiences, decisions, external resources such as time and money, and internal resources such as beliefs and reasons for choices, aligned well with the data. Both researchers discussed difficult passages to code and refined the coding scheme. Rethinking our data in this way, we began to conceptualize Andrew's internal resources as reflecting his perceptions and beliefs about his experiences. Consistent with the conceptual framework, the final coding resulted in using the themes of experiences (Figure 1, Box 5), decisions (Figure 1, Box 2), feelings or perceptions (Figure 1, Box 1) and other. These themes were chosen because they were the most dominant in Andrew's narrative. Finally the researchers looked for any discernable patterns to help in delineating Andrew's responses relative to the research question. The codes of experiences and feelings or perceptions were well mixed throughout the discussion of Andrew's teaching practices. When he discussed his teaching, he frequently reflected on his personal learning experiences. Due to limited space, findings will be shared through Andrew's quotes as well as brief summaries that were developed through the four cycles of coding.

Findings

Pedagogically Influential Life Experiences

"I know what it's like to be in that position [lost in class] and I don't want any student to ever feel like that… it's just now a permanent perspective for me." The analysis showed that Andrew's life experiences influenced his teaching through his dispositions and attitudes toward teaching. The three final categories used in the coding scheme (experiences, decisions, and feelings/perceptions) were well mixed throughout the interview. However, once the interview focused solely on his teaching, a pattern was observed. The feelings/perceptions codes had been further broken into opinions, attitudes and dispositions. While discussing his teaching, these codes became heavily concentrated on dispositions (about 55%) and attitudes (about 25%). Prior to this, feelings/perception codes had been coded as opinions two-thirds of the time. The code of experiences consistently appeared throughout discussion of teaching in the last five interview questions, indicating a close connection between his personal experiences and thoughts about teaching his students. The classroom environments Andrew experienced while in school and his experiences with failure were themes that emerged from the data analysis as influential in Andrew's teaching. Each experience will be described below.

Classroom environment: Class needs to be interesting. As a teacher, Andrew acted, told stories, and used spontaneity because in his experiences as a student, he was driven to work when he found the concepts to be interesting.

> If you do something else like act weird and put a little bit of theater into it, students who maybe are not intrinsically motivated will … pay attention to you, and then hopefully they'll start paying attention to the material, and maybe they'll see something that their interested in…You gotta make it interesting in some way…You can't get somebody to think it's (mathematical content) interesting without being interesting yourself.

This comment, coded as feeling/perception, immediately followed Andrew reflecting on his student experience. We began to look back at the codes for Andrew's high school and undergraduate experiences and saw that Andrew worked to put effort into what intrinsically interested him. Furthermore, when Andrew didn't like the way a class was taught, he wouldn't do the work. "When I have intrinsic interest, I'm able to do any problem that's given to me. But when I'm not intrinsically interested, it's really hard for me to sit down and focus." He viewed classes where a teacher just writes what the book is saying on the board as a waste of his time, and would “rather just go read the book on my own time and not sit in this room and get the same information.”

Andrew wanted his class to be interesting for his students so the students would pay attention and engage with mathematics. While most, if not all teachers would say they want their class to be interesting for their students, Andrew’s attitude can be linked to his personal preferences as a student. This is an example of what Connelly & Clandinin (2000) call teachers’ personal practical knowledge and is consistent with research concluding that life history influences teaching (Bukor, 2013).

Experiences with failure: Start with the basics and don’t judge. Andrew’s school experiences included not being able to finish work, being dropped into lower academic tracks, dropping out of high school, being older than other students, beginning collegiate math at the lowest level, seeing teachers’ writing on board and having no idea what’s happening, and choosing to give up and not learn. Our codes of feeling/perceptions about teaching were frequently immediately preceded by or followed by portions of the interview which were coded as experiences. Andrew found himself at a perceived disadvantage as he experienced failure in school. So, he related to students who struggled with mathematics and wanted to help them find academic success. Andrew believed that good teaching meant filling gaps in student understanding, and directly taught his students to use metacognition when solving problems in class. Moreover, Andrew empathized with students who struggle in school. He decided not to judge or belittle students because of his own difficult experiences in school.

Effects of Pedagogically Influential Life Experiences.

Andrew’s life experiences influenced his teaching practice in how he chose to present material. Both his presentation style and instructional design are linked to his personal biography. Andrew actively worked to keep his students’ attention by performing, telling stories, acting silly, mixing up the routine, and creating a casual classroom atmosphere. His instruction was very guided, logical, and step-by-step. He viewed “doing” mathematics as a process and discussed metacognition with his students.

The four cycles of coding showed that Andrew used his experiences as a filter to look at his own students. His biography further influenced his teaching practice through the relationships he sought to build with his students. He was open to dialogue, willing to offer help, and wanted his students to participate and answer questions during class, not just lecture to them. His memories of how it felt to be lost or to give up made him empathetic, led him to give students the benefit of the doubt, and conditioned him to look for students who struggle. "I'll look for the students that are doing the worst in the class" because he doesn't want them to have the same adverse experiences he had with school.

Discussion

In answering our research question, we identified that Andrew drew on his classroom environment and his experiences with failure as primary influences that affected his teaching of undergraduates. Moreover, when analyzing how he drew on such experiences, Andrew often used these experiences as a lens for avoiding teaching decisions that would result in situations similar to his negative past experiences, while promoting those decisions that recreated his positive learning experiences. Referring back to the theoretical framework (Borko et al.,1992), Andrew's experiences (Figure 1, Box 5) influenced both his feelings and perceptions about teaching (Figure 1, Box 1) and his teaching decisions (Figure 1, Box 2). More specifically, our analysis showed that his dispositions and attitudes toward teaching were grounded in his personal experiences and that these feelings informed his classroom-level decisions.

This study is only a single case study, yet it offers a valuable counter-narrative to the traditional mathematics educator who can successfully "do mathematics" while also examining how personal biography influences teaching practice. Andrew experienced high levels of academic failure yet still became a mathematics doctoral student, reminding us that the pursuit for knowledge is not a race. Efforts to recruit more STEM majors may benefit from broadening their focus to include students such as Andrew, who take less traditional paths into their fields. Andrew expressed concerns over the lack of opportunities, such as internships, offered to transfer students. STEM educators could expand their reach if they could encourage marginalized students such as Andrew.

To understand teacher decision-making, researchers need to understand why teachers act, think, and respond in particular ways to student learning (Bukor, 2013). Andrew's counter-narrative seeks to add to this understanding of how teachers' personal experiences contribute to their beliefs about teaching and classroom practices. This understanding may be useful to teacher educators, those designing professional development for in-service teachers, or those working with developing graduate teaching assistants. It reminds teachers that learning mathematics is not a race and may prompt them to reflect on their own personal experiences.

We face a research paradox in mathematics teacher education (Ko, Yee, Bleiler-Baxter, & Boyle, 2016) where we desire to produce teachers with strong content knowledge of mathematics, but we also desire for teachers to have the experience of the *productive struggle* (Warshauer, 2011) within mathematics so that they are able to relate to their students' struggles. Our results showed that Andrew's attitudes and dispositions towards teaching were grounded in his personal experiences; we see his empathy and caring for his students' also stems from his struggles. Thus while Andrew is a counter-narrative with respect to his life's journey, he can be an ideal candidate for teacher education.

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BALANCING TEACHERS' GOALS AND STUDENTS' PLAY IN A SANDBOX-STYLE GAME ENVIRONMENT

Justin Seventko
Montclair State University
Seventkoj1@montclair.edu

Nicole Panorkou
Montclair State University
Panorkoun@montclair.edu

Steven Greenstein
Montclair State University
Greensteins@montclair.edu

Sandbox style video games, such as Minecraft, are an increasingly popular technology in classrooms. However, when such gaming environments are used during mathematics instruction, teachers' instructional goals are oftentimes at odds with students' agency to pursue self-directed activity. By acknowledging that agency is a powerful force that should be honored but perhaps also negotiated, this research seeks to identify the essential elements of the design and implementation of tasks that generate a space for mathematical activity in which both teachers and students are afforded agency and have the capacity to exercise it.

Background: The Relationship between Agency and Gaming

At its essence, agency can be thought of as the capacity for an individual to make choices based on their own free will without constraints placed on them by external factors or structures, such as social class, gender and ethnicity. Within the classroom environment, agency can manifest itself in a variety of ways. Relational agency, for instance, refers to the ability to "engage with the dispositions of others" (Edwards & D'arcy, 2004, p. 147). By flipping the roles of students and teachers in a bilingual classroom, Edwards and D'arcy (2004) found changes in both the power relationship between students and teachers, as well as the very way students thought about language competency. Agency can also be thought of in terms of changes to what it means to be a member of a classroom (Rainio, 2008). A teacher may, for instance, choose to give up managerial control of their classroom to students in order for them to manage their own learning. Such student-centered approaches and loosening of classroom control may enable students to manage their own learning through opportunities to teach their fellow classmates or engage in collaborative research.

Sandbox Style Games as a Space for Agency Creation

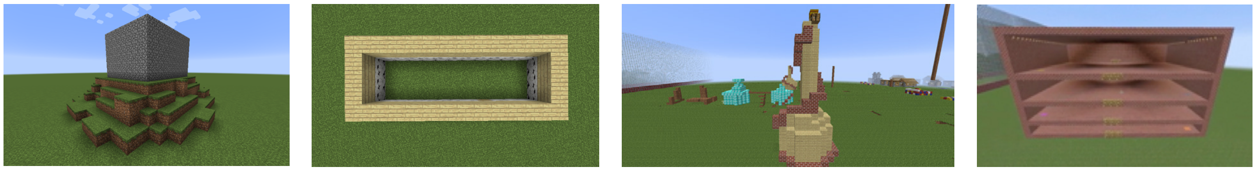
One might wonder what resources might be used in the mathematics classroom to design and implement tasks that not only teach content but also create an agentive space for students. Due to the inherently open nature of their game play, one potential solution may be "sandbox style" video games, which typically provide users with "tools to create their own environment and goals" (Williams, 2010, p. 15). Minecraft is an open-space environment with graphics that are purposefully "blocky" and provide a "visual allusion to LEGO™ " that "suggests a space in which the player is given free rein to create whatever he or she wishes from the pieces provided" (Duncan, 2011). Players use different types of blocks and tools to build structures of their choosing. There is no sense of "winning" and "losing" because players pursue goals that they themselves set such as exploring a certain part of the world or building a structure they designed.

While the potential may exist for sandbox style games to create an agentive space in the mathematics classroom, there has been little research into how to best design tasks for such environments and implement them in the classroom. A survey study by Takayama (2008) found that 74% of K-8 teachers use digital games in their instruction, and as this number continues to increase, it is necessary to research the ways in which teachers can delicately balance a variety of obligations that greatly influence the design of tasks and how they are implemented (Herbst, 2003). An example is understanding how to achieve a balance between the instructional goals of the teacher and the goals of students' self-directed activity when such gaming environments are utilized in the classroom (Hoyles & Noss, 1992). This conflict of goals has been deemed the "play paradox" (Hoyles & Noss, 1992, p. 47): the ways students explore and solve a problem may not lead to the mathematics content the teacher planned or desired. Even if "mathematical nuggets" are carefully planted, the students may still ignore them and pursue an avenue of their interest rather than the intended content.

With these concerns in mind, this research seeks to address the following question: What are the characteristics of sandbox-style mathematics tasks that honor both students' agency to pursue self-directed activity and teachers' agency to pursue instructional goals?

Methods

We use a design-based research methodology (Barab & Squire, 2004; Brown, 1992; Collins, Joseph & Bielaczyc, 2004) aiming to understand the "learning ecology" (Cobb, Confrey, Lehrer, & Chauble, 2003) of Minecraft as a space for agency and mathematical activity. We designed an initial set of four tasks (Figure 1) for use in Minecraft by drawing on students' everyday experiences with building and making and exploring existing mathematical tasks designed for Minecraft. Our tasks were designed using the task design features of Ainley, Pratt, and Hansen (2006) and Stein, Grover, and Henningsen (1996), such as emphasizing the purpose and utility of mathematics and allowing multiple-solution strategies.



Stone Pile task House Building task Goat Tower task Staircase task

Figure 1. Screenshots from the four tasks.

Each of the four tasks pictured above was contextualized as revitalizing an imaginary world named “CraftLand” and embedded with mathematics content. The Stone Pile task asked students to find the volume of a rather large pile of stone. The House Building task placed a supply restriction on students: they were tasked to build a house for CraftLand that had a base perimeter of 36 blocks. The Goat Tower task asked students to consider out how they might be able to create a circular-based structure in the otherwise “blocky” gaming environment and examine how scaling the tower to a larger height would impact the number of stairs that wrapped around it. The final task, Staircase, presented students with a multilevel structure without any stairs. Their objective was to create stairs between the floors that went from one spot indicated by a color to another colored spot in such a way that minimized the blocks used.

We conducted teaching experiments (TE) (Confrey & Lachance, 2000) in order to modify and refine this set of tasks. The TE consisted of ten one-hour sessions, with a small group of five students (four 4th grade students and one 5th grade student). These students were enrolled in a STEM summer camp and were chosen because they had worked with Minecraft before and were familiar with the environment. While the tasks presented did not require previous experience with Minecraft in order to be successfully completed, we deliberately chose students with experience to minimize time spent explaining the gaming environment. The primary data source was video recordings of both classroom discourse and computer interactions.

Analysis and Coding

Ongoing analysis began with the formulation of research-based initial conjectures before the TE and these evolved throughout the duration of the design study (Cobb et al., 2003). In this paper, we focus on the retrospective analysis we conducted at the end of the TE aiming to capture the extent to which the task design generated a space for agency. The unit of our analysis was the *interactions* of the students within the Minecraft space. To describe the nature of these interactions, we used Rainio’s (2008) framework for analyzing the development of agency empirically which identifies three types of students’ actions in a learning environment (Table 1).

Table 1

Distinguishing between Interactions and Non-interactions

Classification	Type of action	Evidence by giving examples from the data
Interaction	**Initiative:** Student shapes the flow of the gaming activity for not just the student who took the action, but others in the classroom and gaming environment as well.	Students ignoring the teacher's intended learning task and instead exploring the world on their own and completing tasks they created on their own.
Non- interaction	**Responsive:** Student follows typical classroom norms.	Everyday classroom actions, such as answering a question posed by the teacher, following instructions or performing an action when asked (such as sharing a strategy to solving a problem).
	Passive: Student is not responding to the actions of those around them, whether in the gaming environment or in the classroom.	When a student, John, proceeded to count the number of blocks in a stone pile without asking questions or interacting with other students, he was acting passively.

We share Rainio's (2008) view that "even though being passive and responsive can produce some agency for the participant in the context of their being students in school, it does not reveal anything about the student's agency in the local emerging activity" (pp. 123–124). Consequently, an interaction was identified *only* if it showed an initiative action. After delimitating the data into interactions and non-interactions, the next step was to characterize the types of interactions. Ranio describes four main types of initiative actions (interactions): supporting, constructing, deconstructing, and resisting (Table 2).

The first two, constructing and supporting, are indicators that an agentive space has been created, that is, students are asked to do things and willing to do them. It is worth mentioning here that in some cases, there were interactions that followed these constructing initiatives that fell into the category of supporting initiatives. For instance, many times, either while in the process of carrying out their strategy or after they successfully completed it, they supported their fellow students who were unable to solve the task by explaining what they did and why. The remaining two types of initiatives, deconstructing and resisting, provide evidence for disengagement, or a lack of agency. During the retrospective analysis, we first identified the interactions in the TE, and then sorted those interactions into the above categories. The count of interactions, as well as the count of the type of interactions, showed the degree in which the task design generated a space of agency.

Table 2

Distinguishing between the Types of Interactions

Degree of Agency	Type of interaction	Evidence by giving examples from the data
Indicators that an agentive space has been created; students are asked to do things and willing to do them.	**Supporting**: Interactions where students assisted each other with an issue or provided some sort of evidence-based or verbal support for a potential solution to a problem.	Evidence of a community of mutual trust between students, such as seen in this exchange between two students, John and Abe, and the instructor: John: I'm stuck in a well. Seriously. Instructor: You are? John: Yes. Abe, can you help me? Abe: I'm helping you.
	Constructing: Interactions where a student developed some sort of new strategy or participated in an event that actively contributed to the activity and changed the flow of it for the other students.	Evidence of various approaches that students took to solve a task, such as finding the volume of a big stone pile. Two examples of this include John electing to dig through the pile to ensure it was solid, and Abe building a structure nearby as "stairs" to get a different view.
Indicators for disengagement, or a lack of agency.	**Deconstructing:** Interactions aiming to disrupt the flow of the gaming environment.	Evidence of disrupting behavior, for instance, while students were working on the stone pile task in the Minecraft environment, John wandered off into the world, away from the task at hand, and found a chest with tools meant to be shared with his classmates. He refused to share these tools with his classmates, thus disrupting the flow of the game for the other students.
	Resisting: Interactions where the student acts in defiance of the power holder, mainly the teacher.	Evidence of students ignoring the teacher's intended learning task and instead exploring the world on their own and completing tasks they created on their own, for example: Instructor: Did you girls read the instructions? Carla: No. Maya: No, it has to be a circle. Instructor: It has to be a circle, right? So, how are you going to build a circle? Carla: We are making a rectangle.

Findings

The two tasks that seemed to generate the greatest space for agency were the Stone Pile Task and House Task, as evidenced by the greater number of Supporting and Constructing Interactions (Table 3). Deeper analysis of video data identified specific characteristics of both tasks that contributed to the creation of an agentive space. In both of these tasks, students were presented with an authentic situation which allowed for multiple paths to a solution. Of particular note is the comparatively small number of Resisting Interactions for the Stone Pile Task and correspondingly higher number of Constructing Interactions. In fact, of all of the tasks created, we saw, by far, the greatest variety of approaches for this task, demonstrating a high-level of student engagement and high potential for students to shape the flow of the activity (Stroup, Ares, & Hurford, 2005). The approaches varied widely, ranging from simply counting the blocks on one layer and multiplying by the height, to digging inside the pile to confirm that it was solid, to building additional nearby structures to gain a perspective on the pile from above. We therefore conjecture that perhaps the Minecraft gaming environment may be particularly suitable to tasks with multiple pathways and an agreed upon endpoint (Stroup et al., 2005)

Table 3

Interaction Classification for Tasks

Task Name	Supporting	Constructing	Deconstructing	Resisting
Stone Pile	3	3	1	1
House Building	4	1	2	3
Goat Tower	2	1	1	4
Staircase	2	1	0	3

Interestingly, regardless of the ultimate amount of agentive space created, we saw a significant number of Supporting Interactions, mainly students engaging in self-initiated conversations with each other about their various solution paths. For instance, the following interaction occurred while working on the Goat Tower task, a task that saw a rather large number of Resisting Interactions and a relatively lower number of Supporting and Constructing Interactions compared to the other tasks:

> **Abe**: O.K. So, [Maya] you have to make all the sides even, which you didn't do. (….) Here, I will draw it out for you. (…) [Draws a pattern on the board] So, go like this and that and that and that and that and that and that.
>
> [Maya attempts to follow Abe's solution]
>
> **Abe**: That's not even, at all. This isn't even, Maya.

This short exchange demonstrates the potential tasks within the Minecraft environment may have in developing a sense of community. Not only did Abe feel comfortable enough to voluntarily share his proposed solution, but he also followed-up with his fellow student to ensure they were on the right track. In turn, the solution he presented shaped the activity for entire class. One of our humble theories (Cobb et al., 2003) is that the ability to easily integrate tasks with multiple pathways to an agreed upon endpoint may promote a sense of collegiality among students and lead to the types of supportive and constructive interactions demonstrated here.

With that said, even interactions that fell into the Deconstructing and Resisting categories provided some significant lessons. For instance, we observed that students were not as responsive to Supportive or Constructive Interactions when they were initiated by a teacher. The idea of their own, personal Minecraft time was sometimes at odds with requests made to them by the facilitators. Most commonly this played out when students refused to engage in a conversation with the facilitator about how they came to their solution. For example,

Facilitator: How did you do that? [constructing a tower during the Goat Tower task]

Abe: You just do diagonal.

Facilitator: How did you do that? What do you mean by diagonal? […] You are not sharing your secret, Abe?

Abe: No.

After seeing similar instances across all tasks, we conjecture that tasks in the Minecraft environment must be designed in such a way that blurs the distinctions students might make between their expectations for the kind of engagement they enjoy at home and the kind they expect at school. Students see it as a violation of their agency when they are asked to leave the gaming environment to explain themselves, thus losing their precious game time. To limit the frequency of these interruptions, we envision features of tasks that develop a student's investment in a well-developed in-game character who frequently must explain his or her reasoning within the game. For instance, immersing students in an engaging, coherent storyline where it is natural that they explain their reasoning may encourage them to transfer that tendency outside of the gaming environment and into other more formal mathematical experiences. In this way, students should begin to think of their time in this role as agentive "Minecraft time" and not as a task that tends toward alienation.

Conclusion

The results of this ongoing research demonstrate that the Minecraft environment may allow teachers to create tasks with certain characteristics that honor both student and teacher agency. As the literature shows, an increasing percentage of teachers wish to integrate games into their teaching, and there is a significant gap that exists in identifying successful task design principles and implementation for gaming environments that aim to respect both the teacher's instructional goals and students' agency to pursue self-directed activity. Going forward, the findings presented here serve as a guide towards revising and expanding this original set of tasks. The various conjectures presented here will be tested and correspondingly revised based on the findings from future iterations of these tasks. The ultimate goal is to conduct several of these teaching experiments in order to develop a Sandbox-Style Mathematical Tasks framework. This framework of both task design and implementation will be built based on the features of the tasks that showed to have honored the agency of both students and teacher.

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AN INVESTIGATION OF STUDENTS’ PERCEPTIONS OF DOING MATHEMATICS

Corrinne Sullivan
Bowling Green State University
sullivc@bgsu.edu

Gabriel Matney
Bowling Green State University
gmatney@bgsu.edu

Jack Jackson II
University of Arkansas – Fort Smith
Jack.jackson@uafs.edu

Garnering different kinds of data from students about their perceptions of mathematics helps teachers, teacher leaders, districts and researchers better understand students’ perceptions. In this study, we investigate and compare students’ perceptions of doing mathematics from samples of students from the United States, China, and Fiji. We administered the Draw Yourself Doing Mathematics instrument developed by Bachman, Berezay, & Tripp (2016) to students at three grade levels in China, Fiji, and the United States of America. Statistically significant differences among perceptions in the three countries and the three grade levels were observed.

At a very young age children are encouraged to draw in order to develop their fine motor skills, stimulate their brains, and cultivate their creativity. Whether drawing lines and circles or drawing a picture of where a child lives, each picture tells the viewer something about the child (Farland-Smith, 2012). Borthwick (2011) shares that “psychologists and art therapists have used drawing for years as a way of gathering information about emotional and psychological aspects of children” (p. 38). As upper elementary, middle, and high school students are still developing their vocabularies and means of expression, using drawings to empathize and gage their perception of a situation can be very effective (Aguilar, Rosas, Zavaleta, Romo-Vázquez, 2016, Finson, Beaver, & Cramond, 1995, Weber & Mitchell, 1996). In support of this assertion, Briell, Elen, Depaepe, & Clarebout (2010) state, “drawings may provide a unique and valuable route of expression even for the older participant who might find it difficult to express such abstract beliefs in verbal or written words articulately” (p. 662) showing that drawing is a valuable tool to gain insights into students’ worlds in the all grades.

In addition to drawings being viable tools for assessing students of varying ages they have also been used to inform researches about students’ perceptions across cultures. Several studies have been done internationally with students as participants. Some examples include Mexico (Aguilar et al., 2014), England (Borthwick, 2011), Belgium (Briell et al., 2010), Canada and Australia (Chamber, 1983), as well as Finland and Russia (Räty, Komulainen, Skorokhodova, Kolesnikov, & Hämäläinen, 2011). However, only two of these studies compared drawings across cultures. Räty et al. (2011), comparing students’ drawings of intelligence in Finland and Russia, found “cross-nationally shared” (p. 17) elements. Similarly, when comparing the drawings of French speaking versus English speaking Canadian students, Chambers (1983) found the drawings to be “very much alike” (p. 262). Therefore, drawings can be a good source of data for exploring perceptions across cultural lines. In this study, we used drawings to investigate the following research questions:

RQ1: What are the differences, if any, among students in the same grade level in the United States, Fiji, and China in perceptions of doing mathematics as measured by the "Draw Yourself Doing Mathematics" instrument?

RQ2: What are the differences, if any, among students from different grade levels from the same country in perceptions of doing mathematics as measured by the "Draw Yourself Doing Mathematics" instrument?

Related Literature

The study presented here further develops Bachman, Berezay, & Tripp’s (2016) *Draw Yourself Doing Mathematics Test* in which students enrolled in a traditional introductory collegiate mathematics course as well as students enrolled in a course pairing mathematics and dance completed drawings at the beginning and conclusion of the semester. The samples were openly coded for affective elements indicating students’ perceptions of doing mathematics. Numerical values were assigned to these open codes which were used to score each sample. Bachman et al.’s (2016) results comparing pre and post test scores of the students between classes showed the course to be effective.

The *Draw Yourself Doing Mathematics Test* heavily relied on the work of Chambers’ (1983) and Finson et. al. (1995) *Draw a Scientist Test* assessing children’s stereotypical beliefs of scientists by asking them to simply draw what they believed a scientist looked like. Farland-Smith’s (2012) *Development and Field Test of the Modified Draw-a-Scientist Test and Draw-a-Scientist Rubric* extended Finson et. al.’s (1995) research by combining the drawings aspect with an additional set of questions asking for additional information about a student’s drawing. This additional information eased the scoring process for the appearance, location, and activity categories.

Research involving students’ drawings has been extended into Science, Technology, Engineering, and Mathematics (STEM) fields since the work of Chambers (1983). For example, Thomas, Colston, Ley, DeVore-Wedding, Hawley, Utley, & Ivey (2016) developed a rubric for assessing fourth and fifth grade students’ knowledge and understanding about the work of an engineer. Some extensions into the branch of mathematics parallel Chambers (1983) and Finson et. al. (1995) such as assessing high school students' and adults' images of mathematicians (Aguilar et al., 2016; Rensaa 2006). However, others diverged from the original test extending the applications of drawing to include assessing the affective elements present as collegiate students draw themselves doing mathematics (Bachman et al., 2016), primary students' perceptions of and attitudes towards their mathematics lessons (Borthwick, 2011), as well as preservice teachers' mental models of mathematicians doing math (Wescoatt, 2016).

Using drawings as a data source has also been extended more generally in education (Briell et al., 2010; Räty et al., 2011; Weber et al., 1996). Drawings are a way to allow students to naturally express their perceptions of experience that involve learning and growing in a new knowledge, such as mathematics. As a data source, drawings are considered to be similar to text and frequently coded in the same way text is coded (Weber et al., 1996). For this reason, we chose drawings as a way to inquire about students' perceptions of doing mathematics.

Methodology

Participants

This study took place in three different countries: The United States of America, China, and Fiji. The participants were students from grades 5, 8, and/or 10/11 who were taking mathematics courses in that grade. Table 1 shows the number of participants from each country and their respective grade levels. Each participant submitted only one drawing.

Procedure

Drawing upon the work of Bachman et al. (2016) we gave the participants the prompt "Draw yourself doing mathematics. Don't worry about the quality of your drawing. Just sketch what comes to mind." The authors partnered with teachers who wanted to better understand their students' perceptions of doing mathematics. With the oversight of the first author, the teachers administered the prompt to their students without a time limit. Teachers distributing the assessment in all countries were instructed that all samples should remain anonymous. Teachers were instructed to inform participants that they were able to include words to explain their drawings, but that a drawing must be present. Following implementation, drawings were collected and numbered. Any drawings that did not have a viable sketch were thrown out to prevent bias in the analysis. Such drawings included those that did not have any people, usually only math figures, in them. Once all drawings were numbered and vetted, the rubric was applied to the remaining drawings.

The first two authors of this research are native USA citizens and have studied the education systems and cultures of China and Fiji. Their study of these systems included travelling to China and Fiji, interacting with students, teachers, and education professors as well as visiting schools. Unlike the USA and Fiji, China does not have English as its primary language. Therefore, prior to coding, drawings from China which contained any language or symbols other than English were interpreted by two linguistic and cultural experts. Both of these experts are native Chinese, have lived in both the United States and China, and speak both Chinese and English fluently. The text in these drawings was translated into English and any cultural references were explained to the first two authors.

To ensure fidelity of rubric coding, the first two authors conducted meetings for the purpose of establishing within-group interrater agreement. The first two authors independently coded 10.40% of the data (31 of 298 drawings) with samples that were chosen using a random number generator. The expected minimum for interrater agreement is $r_{wg} = .9$ (James, Demaree, & Wolf, 1993). Interrater agreement exceeded this minimum as $r_{wg} =.9355$.

Instrument

The *Draw Yourself Doing Mathematics Rubric* was adapted from the coding process of Bachman et al. (2016). This rubric uses a seven point Likert scale to assign a numerical value to each drawing. These numerical values also have corresponding categorical values: severely negative, negative, unpleasant, neutral, pleasant, positive, and extremely positive. The assignment of a specific numerical and categorical value is determined on a set number of positive and negative components within the drawing. The presence of negative components, Confusion, frustrations, overwhelmed, question marks, frowns, etc., correspond to lower scores of three or two. Expletives, statements of hate or other intense negative emotions or actions acquire the lowest possible score of one. The presence of positive components, smile, positive thought bubble, indication of understanding, etc., receive scores of five or six depending on the frequency of the components. Similarly, elations, statements of love, and other intense positive emotions or actions, receive the highest score of seven. Through these categorical values, we establish the degree that participants positively or negatively perceive doing mathematics. Evidence for the numerical and categorical value of the drawing is recorded as well as any additional comments pertinent to the sample. The above mentioned criteria were analyzed within and across the three different countries.

Data Analysis

Excel was used to produce relative frequency histograms of rubric scores for each of the six classes. Minitab was used to compute basic descriptive statistics for each class including sample size, mean, median, interquartile range, and standard deviation. 95% confidence intervals for the means and medians were also computed.

RQ1 and RQ2 are testing for evidence of a higher average positive perception level in a specific class than in another versus a null hypothesis of no difference. Therefore, we are using a series of one-tailed two sample tests. Since the underlying distribution of scores is inherently ordinal in nature, the non-parametric Mann-Whitney test for difference in median was used as the primary test. The Mann-Whitney test does not have any normality assumptions on the underlying distribution. However, since all but one of the subgroups have sample sizes larger than 30 and the distributions of individual scores were examined to be mound shaped, the distribution of mean rubric scores is close enough to a normal distribution to be approximated well by a normal curve. This satisfies the assumptions for a *t*-test, thus one sample *t*-tests were used to provide corroborating evidence of a positive difference in mean. An alpha level of .05 was used for all hypothesis tests.

Results

Summary statistics for each class are tabulated in Table 1.

Table 1: Summary Statistics for Rubric Scores by Country and Grade Level

Country	United States			Fiji		China
Grade	5	8	10-11	5	8	8
Sample Size	18	52	44	39	37	108
Median	4.5	4.0	4.0	4.0	4.0	3.0
	[4.0, 5.0]	[4.0, 5.0]	[4.0, 4.0]	[4.0, 5.0]	[3.0, 4.0]	[3.0, 4.0]
Mean	4.6	4.3	3.8	4.2	3.5	3.2
	[4.2, 5.0]	[3.9, 4.6]	[3.4, 4.3]	[3.9, 4.6]	[3.1, 4.0]	[3.0, 3.5]
IQR	1	1	2	1	2	2
Standard Deviation	0.85	1.21	1.61	0.99	1.26	1.35

RQ1: The data provided evidence that for the same grade level, perceptions of doing mathematics in the United State are higher than those in Fiji and those in Fiji are higher than those in China. This is evidenced by the mean scores of both the fifth grade participants: United

States 4.6 and Fiji 4.2, and the eighth grade participants: United States 4.3, Fiji 3.5, China 3.2. However, hypothesis tests had to be performed to determine if these differences were statistically significant. Table 2 provides the *p*-values for tests for significance of differences in center for the four possible pairings of data groups at the same grade level. From the *p*-values from the Mann-Whitney tests, we see that eighth-grade US participants scored significantly higher than their counterparts in either Fiji ($p = 0.004$) or China ($p = 0.000$). US fifth graders vs. Fiji fifth graders produced a *p*-value of 0.112, and Fiji eighth graders vs. China eighth graders produced a *p*-value of 0.098. Therefore, these pairings were not significantly higher at this grade level.

Table 2

One-tailed Hypotheses Tests Pairs of Subgroups

Groups Compared	Mann-Whitney (*p*-value)	*t*-test (*p*-value)
US 5 vs. Fiji 5	0.112	0.099
US 8 vs. Fiji 8	**0.004**	**0.004**
Fiji 8 vs. China 8	0.098	0.112
US 8 vs. China 8	**0.000**	**0.000**

On the other hand, at the eighth-grade level there is enough evidence to support the conclusion that participants in the United States have more positive perceptions of doing mathematics as measured by this instrument than their counterparts in either Fiji or China.

RQ2: Note that in Table 1 there is data from three different data groups from the United States; grade 5, grade 8, grade 10-11. The mean rubric scores for these three data groups are as follows: US 5=4.6, US 8=4.3, US 10-11=3.8. We also have data for two data groups from Fiji; FJ 5=4.2 and FJ=3.5. This, along with a visual inspection of the histograms from each country in Figure 1, suggests that older students' perceptions of doing mathematics were more negative.

We now examine the results of the Mann-Whitney tests found in Table 2 to determine if the data indicates significant support to reach this conclusion. The data gives evidence that US fifth grade participants have significantly more positive perceptions of doing mathematics than US tenth-eleventh grade participants ($p = 0.022$). Fijian fifth grade participants also have significantly more positive perceptions about doing mathematics than the Fijian eighth grade participants ($p = 0.007$). However, the comparisons of US fifth grade participants to US eighth grade participants ($p = 0.188$) and US eighth grade participants to US tenth-eleventh grade participants ($p = 0.053$) is not statistically significant.

In the case of US participants, there is enough evidence to support the conclusion that participants in the fifth grade have more positive perceptions of doing mathematics as measured by this instrument than participants in the tenth-eleventh grade. Similarly, there is enough evidence to support the conclusion that Fijian fifth grade participants are more positive about their perceptions of doing mathematics as measured by this instrument than Fijian eighth grade participants. Although there is some evidence to suggest that US fifth grade participants also have more positive perceptions than eighth grade participants, and US eighth grade participants have more positive perceptions than tenth-eleventh grade participants, the evidence provided here is not strong enough to reach that conclusion.

We note that if two-tailed tests had been used in each of the analyses above, the *p*-values would have been twice as large. However, the analysis of the data would not lead to different conclusions. Similarly had *t*-tests been used instead of Mann-Whitney tests no differences in conclusion would have been reached.

Discussion

The data for the study was gathered from a convenience sample of students at schools which were familiar to the first two authors. These samples are small and not representative of the countries as a whole. Our use of the country names in this article is only for the purpose of categorical necessity and are not used to imply that these results are indicative of students in each nation as a whole. Due to the space limitations of the proceedings we would like to note that we plan the following topics for discussion during the presentation. Further thoughts on data results and its implications about student's perceptions, benefits of the assessment for teachers and teacher leaders, and further study. Additionally, several drawings from each category will be shared during the presentation as it was not feasible to place these pictures in the proceeding. Lastly, we will also share our seven-point Likert scale rubric.

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EARLY GRADE MATHEMATICS ASSESSMENT – SPATIAL REASONING AND RELATIONAL REASONING SUBTASKS

Lindsey Perry
Southern Methodist University
leperry@smu.edu

The purpose of this study was to investigate the construct relevance, construct representation, and age appropriateness of the items developed for the Early Grade Mathematics Assessment Relational Reasoning and Spatial Reasoning subtasks in order to provide validity evidence based on content. A content review was conducted with four experts, and results from the review suggest that there is moderate evidence that the items developed for the subtasks are representative and relevant to the assessed constructs and appropriate for students in Grades 1-3.

Early mathematics knowledge lays critical groundwork for future mathematics success (National Association for the Education of Young Children & National Council of Teachers of Mathematics, 2002) and is also predictive of reading achievement (Duncan et al., 2007) and future socioeconomic status (Ritchie & Bates, 2013). Number sense, including numeric relational reasoning, and spatial sense are often recognized as two of the pillars of early mathematics education (Clements, 2004; National Research Council [NRC], 2009). Both numeric relational reasoning and spatial reasoning are correlated with future mathematics achievement and are foundational to the development of advanced mathematics concepts (Aunio & Niemivirta, 2010; Fennema & Sherman, 1977; McGee, 1979; Nunes et al., 2007). Because of this, assessments should exist to measure students' abilities in these domains to provide educators with valuable information that can be used to intervene and modify instruction.

While numeric relational reasoning and spatial reasoning are critical concepts for early childhood, few assessments exist that focus on measuring these constructs in young children (Perry, 2016). A few early numeracy assessments, such as the Research-Based Early Maths Assessment (REMA) (Clements et al., 2008) and the Preschool Early Numeracy Skills (PENS) Test (Purpura & Lonigan, 2013), include items that assess numeric relational reasoning concepts such as decomposition. However, these assessments include many other concepts and do not focus on numeric relational reasoning. Therefore, specific interpretations cannot be made about students' numeric relational reasoning abilities. While a few spatial reasoning assessments are available for young children (e.g., Picture Rotation Test [Quaiser-Pohl, 2003] and Children's Mental Transformation Task [Levine, Huttenlocher, Taylor, & Langrock, 1999]), the items on these assessments do not fully represent the construct of spatial reasoning. Spatial reasoning includes both spatial visualization and spatial orientation in both two- and three-dimensions (Bishop, 1980) and both should be represented on a spatial reasoning assessment.

Addressing this gap in available assessments, items were developed by RTI International in 2013 with funding from the United States Agency for International Development (USAID) to construct two new subtasks for the Early Grade Mathematics Assessment (EGMA), a Relational Reasoning subtask and a Spatial Reasoning subtask. The EGMA is currently comprised of eight subtasks that focus on number and operations concepts and has been used in 22 countries to determine overall student performance and improve instruction and learning in Grades 1-3 (Platas, Ketterlin-Geller, Brombacher, & Sitabkhan, 2014; RTI International, 2015). Because the new subtasks have the potential to be used worldwide, it is imperative that the validity of the interpretations made using the scores from the new subtasks be evaluated. This study focused on investigating the validity evidence based on test content for the Relational Reasoning and Spatial Reasoning subtasks and had one primary research question: To what degree is the content of the Relational Reasoning and Spatial Reasoning items age appropriate and representative of and relevant to the constructs of numeric relational reasoning and spatial reasoning, as measured by the expert judgment of mathematics education experts?

Related Literature

The most central consideration when developing or evaluating assessments is validity (American Education Research Association, American Psychological Association, & National Council on Measurement in Education, 2014). Validity is not a property of a test itself but instead is the degree to which the interpretations made using a test score are appropriate, meaningful, and useful (Downing & Haladyna, 1997). The intended interpretations made using a test score rest on certain assumptions (Kane, 1992), which must be supported with evidence that can then be evaluated to determine the validity of the interpretation.

For the EGMA Relational Reasoning and Spatial Reasoning subtasks, one of the primary assumptions that links a score on the subtasks to the primary interpretation of determining overall student performance is that the content on the subtasks is age appropriate and representative of and relevant to the constructs of numeric relational reasoning and spatial reasoning. This assumption represents the core of content-related evidence of validity. The content of the items on an assessment should be relevant to and representative of the constructs of interest (Messick, 1989), which means that the items test the construct in important ways

(Haladyna & Rodriguez, 2013) and include all components of the construct. Additionally, the content should be age appropriate to prevent the introduction of construct irrelevant variance.

Multiple rounds of discussion and pre-pilot testing occurred to consider how to best assess numeric relational reasoning and spatial reasoning. A group of mathematics education, international education, and measurement experts suggested possible item types during an expert panel for EGMA (Platas et al., 2014). After a thorough literature and assessment review, two item writers constructed sample items. These items were pilot tested individually to a few students to determine if the items were clear and solicited students' numeric relational reasoning or spatial reasoning abilities. Based on this information, the item types were revised and additional items for each type were developed.

The items developed for the EGMA Relational Reasoning and Spatial Reasoning subtasks were designed to assess the main components of the constructs of numeric relational reasoning and spatial reasoning. Numeric relational reasoning is the ability to mentally analyze relationships between numbers or expressions (Carpenter, Franke, & Levi, 2003; Farrington-Flint, Canobi, Wood, & Faulkner, 2007), which prevents the need for lengthy calculations. Three main item types (see Table 1) were developed to assess the two primary skills used when reasoning relationally, additive composition and number properties.

Two topics are consistently identified as the primary components of spatial reasoning: spatial orientation and spatial visualization (Bishop, 1980; Clements & Battista, 1992; McGee, 1979; NRC, 2009). Two item types were developed to assess spatial reasoning (see Table 2).

Methodology

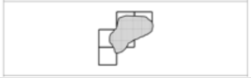
To investigate the construct representation, construct relevance, and age appropriateness of the items developed for the EGMA Relational Reasoning and Spatial Reasoning subtasks, an expert content review was conducted. Four early mathematics education experts reviewed the EGMA Relational Reasoning and Spatial Reasoning items. Reviewers had 10-43 years of experience in mathematics education and research settings, and each holds a Ph.D.

Content review forms were developed by the author of the study and were used to record the judgments of the experts. The expert reviewers completed an Item Review form and an Overall Impressions review form for each subtask. On the Item Review form, expert reviewers rated each item's construct representation, construct relevance, and age appropriateness on a four-point scale (1 = not at all, 2 = somewhat, 3 = mostly, 4 = extremely). Reviewers provided written comments on any ratings of 1 or 2. A four-point scale (1 = not at all, 2 = somewhat, 3 = mostly, 4 = extremely) was also used for the questions on the Overall Impressions review form.

Table 1
EGMA Relational Reasoning Item Types

Item Type		Example Item	Number of Items Developed
Equivalence	Symbolic	$3 + 6 - 6 = \square$	13
	Visual	Read by assessor: "Watch what I am doing. First I am putting 5 balls into the bag. Now I am putting another 4 balls in the bag. Next I take out 4 balls. How many balls are there in the bag now?"	13
Decomposition		$4 + 1 = \square + 2$	13
Multiplicative Thinking	Word Problems	Read by assessor: "There are 3 oranges in each bag. There are 9 oranges. How many bags of oranges are there?"	13
	Visual	Read by assessor: "If 1 circle balances 2 squares, how many squares are needed to balance 3 circles?" 	12

Table 2
EGMA Spatial Reasoning Item Types

Item Type		Example Item	Number of Items Developed
Spatial Visualization	3D	Read by assessor: "Look at these pictures of objects. Please tell me if the two objects are the same or not the same." 	17
	2D	Read by assessor: "Look at these pictures of objects. Please tell me if the two objects are the same or not the same." 	17
Spatial Structuring	3D	Read by assessor: "Look at these pictures of objects. Please tell me how many cubes were used to make this object." 	17
	2D	Read by assessor: "Look at these pictures of shapes made with squares. Some of the squares are covered. How many squares were used to make this shape?" 	13

The expert reviewers' numerical ratings and comments were analyzed to determine if the Relational Reasoning and Spatial Reasoning items represent the constructs of numeric relational reasoning and spatial reasoning and are appropriate for students in Grades 1-3. The median expert rating was calculated for each item for each category/question on each review form. Written comments from the reviewers, if provided, were also examined to determine if common themes were present or if certain items were problematic.

Findings

Relational Reasoning. Based on the expert reviewers' individual item ratings, overall impression ratings, and written comments, there is evidence to suggest that, overall, the EGMA Relational Reasoning items are relevant to and representative of the tested construct and age appropriate. With regard to individual items, 98%, 98%, and 100% of the items had median ratings of 3 or above (i.e., "mostly" or "extremely") for construct representation, construct relevance, and age appropriateness, respectively. The results from the Overall Impressions review form also indicate that the experts, as a whole, rated the items on the Relational Reasoning subtask as representative, relevant, and age appropriate. Additionally, these results indicate that the items are foundational skills for mathematics, assess content predictive of future performance in mathematics, and include content that is teachable and common in early grades mathematics curricula; these criteria are the self-imposed requirements for the content of new and existing EGMA subtasks (Platas et al., 2014).

While the overall ratings were strong, a few of the ratings indicated potential concerns about items or topics, which were further investigated by analyzing the written feedback. Primarily, questions were raised about the strategies students may use to solve the visual equivalence and symbolic equivalence items. For example, the visual equivalence items model addition and subtraction actions with manipulatives instead of numbers or symbols. All four reviewers suggested that children would likely perform mental calculations as actions are shown and completed by the test assessor, rather than employ numeric relational reasoning to determine the number of balls left in the bag. These concerns are also related to the amount of working memory required for these items. Students with better working memory may perform higher on these items, not necessarily because of their numeric relational reasoning ability but because of their working memory ability.

Additionally, two reviewers questioned whether the symbolic equivalence items should be timed to measure fluency (e.g., number correct per minute), which may help determine if students are using numeric relational reasoning or taking time to calculate the entire equation mentally. If the Relational Reasoning items inadvertently lead students to perform calculations instead of relational reasoning strategies, the items may not be assessing the intended construct. A few other concerns were raised about the clarity and familiarity of the language used (e.g., "equation," "expression," "fit") and about whether students have had adequate exposure to concepts such as decomposition and manipulatives such as balances, which may impact the interpretations made using the decomposition items and the multiplicative thinking visual items.

Spatial Reasoning. There is also evidence to suggest that, overall, the EGMA Spatial Reasoning items are relevant to and representative of the tested construct and age appropriate. The experts rated all of the items developed for the EGMA Spatial Reasoning subtask as mostly to extremely age appropriate and representative of and relevant to the construct of spatial reasoning. The ratings on the Overall Impressions review form also mirror these determinations and also indicate that the Spatial Reasoning subtask meets three out of the four self-imposed requirements for EGMA subtasks (Platas et al., 2014), including assessing foundational skills, skills that are predictive of future mathematics performance, and topics that are teachable. However, the reviewers noted that the content of the Spatial Reasoning subtask, while foundational, is not common to many early grades mathematics curricula.

While the items were highly rated, three of the four reviewers noted that the language used in the three-dimensional and two-dimensional spatial visualization items (i.e., the “same” and “not the same”) might be ambiguous and interpreted in different ways by students. Students may have different conceptions of what it means for two objects to be the same: same number of blocks, exact same arrangement (no transformations), reflection of an arrangement, same structure in the same plane, or a rotation or a translation of an object, among others. Distinctions between reflections and rotations may be particularly problematic with this language. For example, 88% of students said that two figures that were mirror images of one another were the same. The ambiguity of the “same” and “not the same” may have contributed to the reviewers rating the instructions for the Spatial Reasoning subtask as somewhat clear and understandable.

Opportunity to learn issues were also raised by a few of the expert reviewers, particularly on the three-dimensional items. Both the three-dimensional spatial visualization and spatial structuring items involve two-dimensional representations of structures built with cubes. One of the reviewers suggested that students may not have previously worked with three-dimensional cubes. Students also may not have seen two-dimensional representations of three-dimensional objects. If students have been exposed to items of this kind, they may outperform other students who have not had similar opportunities. The ambiguity of the language in some of the items and opportunity to learn issues may introduce additional factors to the measure that are unrelated to the assessed construct.

Discussion

Based on the expert reviewers' ratings and comments, there is moderate evidence to suggest that, overall, the items developed for the EGMA Relational Reasoning and Spatial Reasoning subtasks are relevant to and representative of the tested construct and age appropriate. However, some of the experts' comments limit the strength of the evidence for this assumption. Primarily, questions were raised about the strategies students may use to solve a few of the Relational Reasoning item types, potential ambiguity of the language on the spatial visualization items, and whether students have had an adequate opportunity to learn these concepts. Cognitive interviews should be conducted to investigate the cognitive processes and strategies used by students on the Relational Reasoning items. If students are indeed counting or calculating instead of utilizing numeric relational reasoning strategies, the items may not be assessing the intended construct. Additionally, the language on the spatial visualization items (i.e., the "same" and "not the same") should be clarified to prevent confusion. Student interviews could assist in determining how to best modify the item prompt. With additional research and refinement, the items developed for the EGMA Relational Reasoning and Spatial Reasoning subtasks may be promising approaches to assessing students' numeric relational reasoning and spatial reasoning abilities.

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COUNTING SQUARE UNITS IS NOT ENOUGH: EXPLORING AREA DYNAMICALLY

Nicole Panorkou
Montclair State University
panorkoun@montclair.edu

Madhavi Vishnubhotla
Montclair State University
vishnubhotm1@montclair.edu

We explore a new way for students to experience area measurement, what we refer to as Dynamic Measurement (DYME). DYME engages students in dynamic experiences of measuring 2D surfaces by multiplicatively composing two linear measures through 'sweeping.' We describe the design of tasks that we used for engaging students in DYME experiences and report on findings from teaching experiments with six pairs of third-grade students to discuss DYME's potential for developing students' understanding of area as a continuous quantity.

Background and Aims

Previous studies on area measurement focus on covering space with square units and quantifying that covering (e.g. Barrett & Clements, 2003; Clements & Stephan, 2004). The selection of square units to cover a surface does not naturally occur to children (Battista, Clements, Arnoff, Battista & Borrow, 1998; Outhred & Mitchelmore, 2000). Students often leave gaps between units, overlap them, double count them or combine units of different size (e.g. Lehrer, 2003; Battista et al., 1998); even after extensive covering and tiling activities, students still have difficulty drawing square units (Outhred & Mitchelmore, 2000). Furthermore, switching from the one-dimensional approach of counting squares as discrete quantities to the two-dimensional multiplicative relationship of combining two linear continuous measures in an area formula, can be extremely difficult for students (Baturo & Nason, 1996; Kamii & Kysh, 2006). Piaget, Inhelder & Szeminska (1960) argue that these difficulties arise because "the child thinks of the area as a space bounded by a line, that is why he cannot understand how lines produce areas" (p. 350). Both Piaget et al. (1960) and Simon & Blume (1994) suggest that students need to experience area as a continuous quantity in order to develop a conceptual understanding of it. An intuitive way for students to visualize a meaning for area in a dynamic way is to view it as a 'sweep' of a line segment of length *a* over a distance of *b* to produce a rectangle of area *ab* (Confrey, Nguyen, Lee, Corley, Panorkou, & Maloney, 2012) (Figure 1).

Figure 1. Area as 'sweeping" (reproduction from Confrey et al. 2012)

Referring to this dynamic continuous approach to measurement as *Dynamic Measurement* (DYME) distinguishes it from others (i.e. counting units). By considering both length (e.g. of a paint roller) and width (e.g. the rolling distance) as attributes that define an area (e.g. the generated space), DYME emphasizes the role of dimensions (linear measures) in area measurement. Although prior work (e.g. Confrey et al., 2012; Lehrer, Slovin, Dougherty, & Zbiek, 2014) identifies the significance of teaching this dynamic approach, little information exists about how students' DYME reasoning can be developed. Therefore, our goal was to explore: a) What type of tasks and tools may be used for developing students' DYME reasoning? b) What forms of DYME reasoning are visible and can develop as a result of students' systemic engagement in these tasks? c) How and to what extent may DYME thinking support students' development of area as a continuous structure?

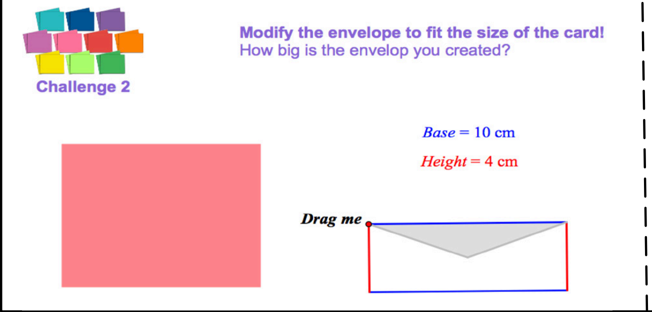
Methods

We used a design-based research methodology (Barab & Squire, 2004) to develop mathematical tasks and tools, focusing on continuous cycles of design, enactment, analysis, and redesign. In formulating initial conjectures about students' reasoning of DYME, we synthesized existing literature on area measurement (e.g. Clements & Sarama, 2009; Confrey et al. 2012) and considered their measurement constructs with the spatial structuring of DYME. For example, to develop the multiplicative relationship of *length x width*, most studies begin with the counting of individual square units, then counting the units in a row (using repeated addition), then counting the units in a row and column and multiplying *rows x columns*. In contrast, the spatial structuring of DYME focuses on visualizing composites of 1-inch paint rollers iteratively dragged over a specific distance to cover a surface. A surface is described in terms of the number of 1-inch swipes (length) and the distance of each swipe (width). For example, to cover a surface of length 4 cm and width 7 cm, we need 7 one-inch swipes of 4 cm. Our design promotes student thinking about commutativity (e.g. 7 horizontal 1-inch swipes of 4 cm is the same as 4 vertical 1-inch swipes of 7 cm) and also reversibility (e.g. constructing surfaces by iteratively dragging 1-inch rollers and deconstructing surfaces of length a and width b by equally splitting the surface into a sections of length 1 and width b to find area). The target understanding of DYME involves a dimensional deconstruction (Duval, 2005), analytically breaking down a 2D shape (its area) into its constituent 1D elements (length and width measures) based on relationships. Thus, the two quantities (length and width) are coordinated simultaneously when making judgments about size.

We used Geometer's Sketchpad (Jackiw, 1995) and its features to design our tasks. In addition to the dragging tool, the trace tool gives a trace of all the points on line segment (paint roller) following a locus as they move on the screen. Our conjecture was that the user would associate this discrete trace with the continuous surface formed. We conducted a series of teaching experiments (TE) (Steffe & Thompson, 2000) to modify and refine DYME materials, monitor effects on student learning and document changes in their DYME reasoning. Twelve third-graders worked in pairs for 6-9 sessions of 45-90 minutes each. The students represented various abilities and all students had some instruction on area as tiling the year before the TE.

Findings

We will describe the learning process demonstrated by two students, Isaac and Lara, using selected tasks and episodes from their TE. At the beginning of the TE, Isaac and Lara were asked to explain length and area and they responded, length is "Times tables. Length times width." Area is "Like if you have a big box, then you find how many boxes are there in it." Upon elaboration, they stated that they "didn't really understand it." In contrast to seeing area as a space consisting of discrete area units, early explorations in DYME begin by seeing area as a continuous quantity. Our early tasks focused on matching shapes by modifying their base and height (Figure 2) and coloring surfaces using paint rollers to make connections between roller length and height as well as between distance of paint and base (Figure 3). Due to the ambiguity of the word 'length' we used the terms 'base' and 'height' in the beginning, and 'length' and 'width' later. Students explored the changes to the size of an object when one or both dimensions changed. By the end of the tasks, students were able to coordinate the two dimensions as attributes to make judgments about size and use the 'base' and 'height' language to describe shapes (e.g. describing the size of a shape as "Its base is 5 cm and its height is 6 cm").



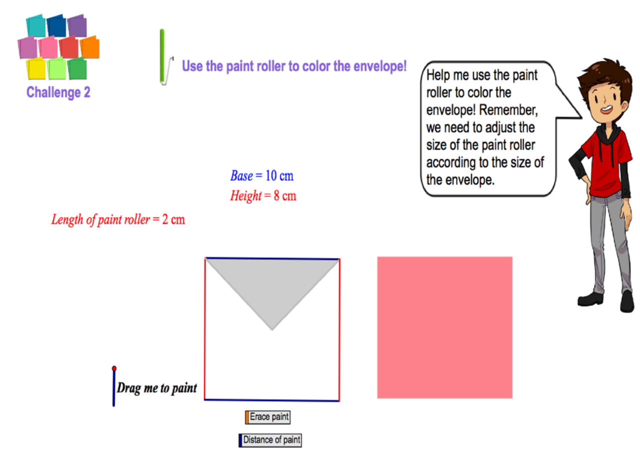
Modify the envelope to fit the size of the card!

How big is the envelop you created?

Researcher (R): So for writing your theory, what needs to change and what needs to remain the same?

Lora (L): The height needs to change and the base needs to remain the same, because it's the same [the base] as the envelope's.

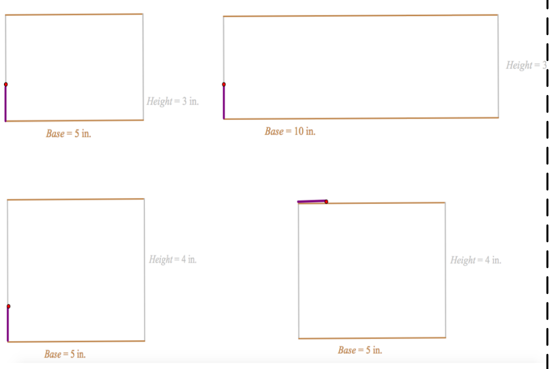
Figure 2. Modifying an envelope to fit the size of the card



R: How big do you need to make the roller?
L and Isaac (I): We need to make it 8cm.
R: How do you know it's 8cm?
L and I: Because the height is 8.
I: If the height is 8, so you are gonna want to make the length of the paint roller 8cm too so that it can match.
R: OK. What do you think would be the distance of paint?
L: I think it's 10 because they said that the base is 10.
I: Because if the base is 10 then the distance of the paint should be 10cm.

Figure 3. Painting and reasoning about the length of the roller and the distance of paint.

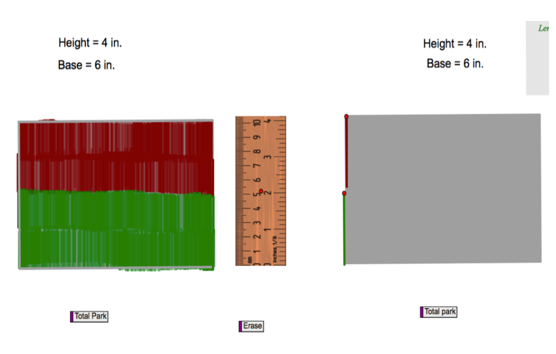
In the next set of tasks, our aim was for students to recognize the multiplicative relationship between the length of a roller (height), the distance of paint (base), and the space covered (area). Students used a 1-inch roller to paint shapes that had the same base but different heights and intuitively began using the multiplicative 'times' language to find the space covered as evidenced in comments such as "we need three swipes of ten" and "we need to do 10 three times" (Figure 4).



R: How much space we will cover in this one? [showing the 10 by 3 wall]
I: I think this is 30.
R: Why?
I: Because the base is 10.
L: Since this is small [the roller] we are going to swipe like that and swipe like that and swipe like that …we are going to add 3 tens
I: You are going to count by tens for three times. [..] And that's the same thing [the fourth shape as the third shape]. It's just that they put it [the roller] there. The difference is that you are going up and down.
R: I don't understand why it's the same.
L: The height is the same it is 4 and the base is 5.
I: The difference is that you are going up and down and here you are going side to side.
L: We are just counting by 4s here. If you add 4 five times then you will get 20 just like that one.

Figure 4. Painting walls using 1-inch rollers and reasoning about the amount of space covered.

Subsequently, they used rollers of different sizes (2-inch, 3-inch, etc.) to paint the same shape and explored the amount of space each roller covered by visualizing each 'large' swipe as a composite of 1-inch swipes (Figure 5). Gradually, students moved from area as painting to area as base times height as they recognized that the height of a shape shows the number of 1-inch swipes and the base shows the rolling distance.



Height = 4 in.
Base = 6 in.

Height = 4 in.
Base = 6 in.

[The first park had 1-inch rollers, while the second park had 2-inch rollers. Students talk about the space of soil in the second park]
I: We can do 6… wait…...6 plus 6… then the distance of soil is going to be the same.
R: Why did you do 6 plus 6?
I: It's 6 plus 6 because the height is 2. So you do 2 times 6.
R: Okay…How big is the total park?
L&I: 24.
R: How do you know?
I: Because that one is 24 (first park) and that one is 24 (second park).
L: They have the same base and the same height.

Figure 5. Using different sized grass rollers and soil rollers to cover two parks (1-inch rollers in the first and 2-inch rollers in the second).

Our next goal was to see if students could recognize length and width measures as factors that determine area. Students were asked to multiplicatively change (e.g. double, halve) the amount of space presented to them and reason about the new dimensions and then the reverse, multiplicatively change one or both dimensions, and reason about area (Figures 6 and 7).

Length of Snow white dress palace = 5 in.
Width of Snow white dress palace = 6 in.
Area *Snow white dress palace* = 30 in^2

Length of Cinderella shoe house = 4 in.
Width of Cinderella shoe house = 2 in.
Area *Cinderella shoe house* = 8 in^2

Length of candy store = 2 in.
Width of candy store = 3 in.
Area *Candy Store* = 6 in^2

1. The width of the Snow White dress palace should be half of what it is now. What would be the area of the Snow White dress palace?
2. How much space are we going to cover using the carpet roller?

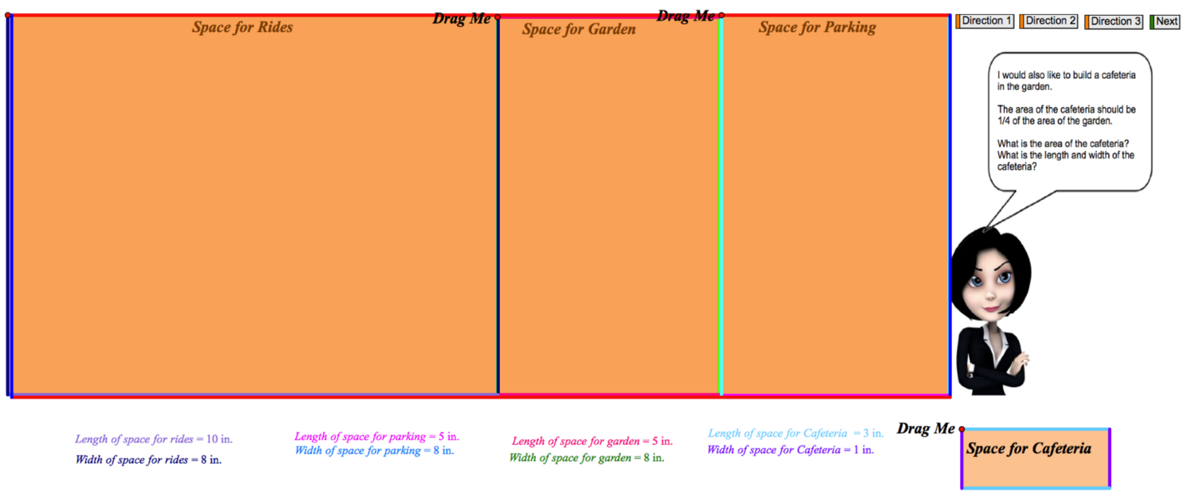
Length of roller = 3 in.
Length of roller = 3 in.

Length of Game Zone = 3 in.
Width of Game Zone = 2 in.
Area *Game Zone* = 6 in^2

Length of toy world = 3 in.
Width of toy world = 2 in.
Area *Toy world* = 6 in^2

R: If you change the width of the Snow White dress palace to half what will happen to the area?
L: It would come down. Probably...
I: I think 15 because if you put 15 plus 15 it will be 30.
L: Half and half.
R: Why did you split it?
I: Because the width is changing and it is going to take half away from what it is now and then if you are going to take a half away the area is going to come in half.

Figure 6. Changing the width and recognizing the effect on area.



I would also like to build a cafeteria in the garden.

The area of the cafeteria should be 1/4 of the area of the garden.

What is the area of the cafeteria? What is the length and width of the cafeteria?

Length of space for rides = 10 in.
Width of space for rides = 8 in.
Length of space for parking = 5 in.
Width of space for parking = 8 in.
Length of space for garden = 5 in.
Width of space for garden = 8 in.
Length of space for Cafeteria = 3 in.
Width of space for Cafeteria = 1 in.

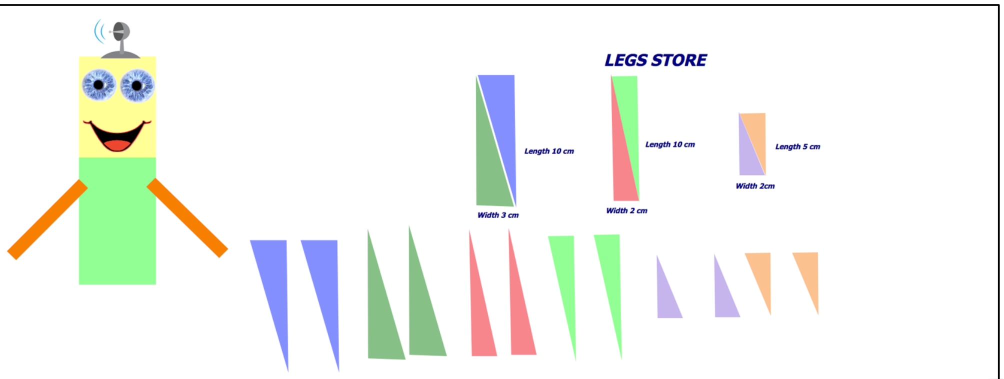
[Students were asked to create a cafeteria with area as ¼ of the garden. Garden's length was 5 and width was 8.]
R: What does one fourth mean?
I: It is a fraction.
L: If we were to split this into 4 parts then one of those parts will be the cafeteria.
R: How big are you going to make it [the cafeteria]?
I: We are going to make it...The length would be 5 inches and the width is gonna be...
L: 2 inches.
I: It would be 2 inches.
L: Yeah because if we use only one-inch roller then it would go 8 times across. But if you use 2-inch roller then it would go 1, 2, 3. And that would go 4 parts.

Figure 7. Changing the area and recognizing the effect on dimensions.

Although students were not asked to paint as they did in previous explorations, we provided paint rollers as a resource. We found that these rollers became a powerful tool to help students visualize the size of covered spaces (see the excerpt in Figure 7) and for transitioning from sweeping-based reasoning to reasoning about area as length *times* width. Students associated the software's discrete trace with the continuous surface that was formed and defined that surface using two linear measures. During the final task (Figure 8), students were asked to coordinate relative areas by splitting rectangles, such as splitting a rectangle in two right triangles.

Significance and Next Steps

This study examined a dynamic way of learning and teaching measurement. Our findings show that students avoided the misconceptions presented in prior research and that they were able to make meaning of the area formula by visualizing area as a continuous, dynamic structure defined by two linear measures: length and width. Among our future goals is to explore how we can make the transformation of units (from linear to area measurement units) more accessible to students who have not been taught powers before. Additionally, we are currently exploring how DYME can be used for extending area measurement to irregular shapes and how this may lead to the notion of the definite integral of the differential calculus. These findings will contribute to research on the teaching and learning of measurement and be useful to both teachers for engaging their students in this type of measurement, and the designers of curriculum and professional development who wish to support the learning of these ideas.



[Students create a robot that has no more than than 150cm^2 area by composing shapes. Here they discuss the area of the orange leg.]

R: What's the area of the orange leg?
I: The whole thing is 10.
R: How do you know?
I: Because 5 times 2 is 10.
R: OK, if you take the purple out [one triangle] how much will it be?
I: 5.
R: How do you know?
I: Because if you had that just alone it would be 5, but if you add another one it will be 10.

Figure 8. Finding the area of a right triangle.

Acknowledgements:

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STUDENTS' UNDERSTANDING OF THE CONCEPTS OF RATES OF CHANGE AND FUNCTIONS

Pragati Bannerjee
Texas A&M University-Corpus Christi
pbannerjee@islander.tamucc.edu

This study reports a cross-sectional research conducted in a public high school with 187 Grades 7-12 students enrolled in Math-7 to Calculus. All participants completed the Diagnostic Test on Rates of Change (DTRC) - an instrument that included focused problems on various representations of rates of change in physical and functional situations. The study found that students' performance on the DTRC test did not show a similar progression in the internalization of the concepts of rates of change, and in formulating mathematical structure of functions as prescribed in the state curriculum standards.

Many processes in the real world are mathematically described by non-linear relations such as polynomial, exponential, logistic, and logarithmic. What is common among these is that the growth rate in the dependent variable is not constant, in contrast to a linear relationship. This study deals with students' inherent difficulty in understanding of functions in general, and their failure to understand what differentiates a linear function from a non-linear function. The purpose of this study was to explore differences and similarities of grade 7 through grade 12 students' understanding of rates of change and functions; specifically, the nature of student difficulties as they reason, represent, and make connections between various representations of rates of change in physical and functional situations involving two co-varying quantities.

It is important that students understand the concept of rates of change to make sense of different types of functions. Muzangwa and Chifamba (2010) observed that students fail to recognize and distinguish between various types of functions when they enter higher math courses such as Algebra II, Precalculus, and Calculus. One of the difficulties that students have is relating the symbolic and graphical representation of rate of change to a written description of a real life situation, and they tend not to pay attention to the units to guide them. Many think of average rate of change as average in the arithmetic sense and carry a misconception that it is possible to calculate instantaneous rate of change exactly from a table of values of a function.

In order for students to gradually build their conceptual understanding of core ideas central to the learning of mathematics, the states have prescribed curriculum standards at each grade level. When students' foundation for the mathematics content knowledge is built on core ideas that are part of the mathematics curriculum, and also become part of teaching practices, then students form a robust understanding of mathematics. Mathematical ideas of proportionality, and rates of change are core ideas for the study of relations and functions. For this study, the rate of change of a function is defined as a ratio that compares the change in the dependent variable to the amount of change in the independent variable; $m = \frac{\Delta y}{\Delta x}$.

Perspective

Carlson, Oehrtman, and Engelke (2010) identified students' reasoning abilities and understandings central to precalculus, and foundational for calculus. These include a strong understanding of rate of change, a process view of function, and the use of covariational reasoning. Some school districts require students to earn credits beyond the state minimum to graduate from high school. Students in these states take Precalculus or Calculus during high school. However, mathematics is not a list of disconnected coursework. Powerful mathematical knowledge results from reasoning with mathematical principles coherent in all these specialized courses and with progressions of learning that leverage these principles, as students build this knowledge over grade levels. The research question for this study, therefore, was: Do students' understandings of rates of change appear to consistently build in a manner that could lead them to develop a solid mathematical structure of functions by the end of Precalculus?

Slope and its various representations are introduced during grades 6-9: all except *trigonometric* and *calculus conceptualizations* indicating a very broad coverage of the concept (Nagle & Moore-Russo, 2014). Grades 7-8 instruction focuses on proportional thinking to prepare students for proportional relationships of the form y=kx and linear relationships of the form y=mx+b. Grade 9 sets the stage for comparing the constant rate of change of linear relationships with the variable rate of change of non-linear (quadratic, exponential etc.) relationships. The student understanding that linearity entails repeated addition whereas exponentiality entails repeated multiplication is a cognitively challenging concept and students take more time to construct this notion in their minds (Ebersbach & Wikening, 2007). Students' success in precalculus course and beyond depends on their understanding covariational characteristics of classes of functions through the *functional property* conception of slope.

Whereas the rate of change between two covarying quantities is at the heart of the functional property conception of slope, students do not necessarily relate increasing and decreasing intervals to rate of change (Hauger, 1997). This situation makes it difficult for teachers to transition to teaching non-linear concepts in the classroom. De Bock (2002) suggested that students' prior knowledge of linearity and proportionality is incompatible when they come across new ideas such as polynomials and non-proportional problems.

Methodology

This study investigated students' understanding of the concepts of rates of change and functions in a public high school in southern part of the United States. The participants were 187 Grades 7-12 students enrolled in Math-7 to Calculus (AB). Student distribution by grade and current mathematics course are presented in Table 1.

Table 1. *Student Distribution by Grade and Current Mathematics Course*

	7	8	9	10	11	12	All
Math 7	17	0	0	0	0	0	17
PreAlgebra	28	0	0	0	0	0	28
Math 8	0	22	0	0	0	0	22
Algebra I	0	18	6	0	0	0	24
Geometry	0	0	13	17	0	0	30
Algebra II	0	0	0	2	11	0	13
C. Algebra	0	0	0	0	0	5	5
Statistics	0	0	0	0	0	3	3
Precalculus	0	0	0	0	4	29	33
Calculus	0	0	0	0	0	12	12
All	45	40	19	19	15	49	187

Note. Rows include participant count by Course and columns include participant count by Grade

All the participants completed a researcher developed DTRC (Bannerjee, 2016). The DTRC construction process began with a careful selection of twenty-six problems based on a very broad coverage of rates of change and its various representations. These problems were either adapted from the textbooks, from Partnership for Assessment of Readiness for College and Careers (PARCC), or written by the researcher. The content validity of the DTRC instrument was established by a review from three expert professors of math education to ensure that the items on the DTRC were appropriate for measuring students' understanding of rates of change and functions. The expert review resulted in 13 problems that focused on rates of change and its various representations in mixed format now used in DTRC. The problems on the DTRC addressed Geometric Ratio, Algebraic Ratio, Measure of Steepness, Functional Property, Rate of

Change in Covariational Context, Parameter in a Linear Function, Real World Situation, Determining Property, and Calculus Conception of slope (Nagle & Moore-Russo, 2014). Figure 1 shows one of the items on the DTRC.

In 2004, there were approximately 275 students in the Delaware High School band. In 2010, that number increased to 305. Find the average rate of change in the number of students in the band from 2004 to 2010.

Figure 1. DTRC Problem #2.

The DTRC was administered to the participants by their classroom teachers. Students were encouraged to show work for each problem on the test booklet. While the use of graphing calculator was prohibited, the participants were allowed to use scientific calculator. The response data from all the participants were scored by the researcher using the scoring guide. With each DTRC problem worth 2 points, the maximum score possible was 26 and minimum score was 0. A preliminary score of 0, 1, or 2 was awarded for each problem based on students' work indicating *no understanding*, *partial understanding*, or *clear understanding* respectively. To establish the reliability of the scoring guide, 20 randomly selected tests were scored by a second rater who had no stake in the study. An inter-rater agreement (Cohen, 1960) was reached among raters. Table 2 displays participant's expected score on the DTRC by math course.

Table 2. *Participant's Expected Scores on DTRC Corresponding to Math Course*.

Math Course (Course Code)	Participant's Expected Score on DTRC
Math-7 (MAS)	≥ 8 points
Math-8 (MAE), PreAlgebra (PAL)	≥ 12 points
Algebra I (ALG)	≥ 16 points
Geometry (GEO)	≥ 18 points
Algebra II (AL2)	≥ 20 points
College Algebra (COA), AP Statistics (STA), Precalculus (PCA)	≥ 20 points
AP-Calculus AB (CAL)	=26 points

Data Analysis

Descriptive statistics were calculated to analyze how DTRC performance varied by grade and by current mathematics course. To verify if there is a significant difference in the mean scores by grades or by math courses, Welch's Test for Analysis of Variance (One-way ANOVA) including Games Howell Pairwise Comparison and a Simultaneous Test for Difference of Means were conducted. Before performing One-way ANOVA, it was verified that sample size for each grade level was ≥ 15. The standard deviations for total scores for each grade level were different. To verify if these differences were significant, a test for equal variances was performed on total scores versus grade level. The result indicated that standard deviations of total scores at each grade level were significantly different [i.e., p-value (0.002) < α (.05)]. Consequently, Welch's Test for One-way ANOVA was employed instead of Fischer's One-way ANOVA that assumes equal variances. Games-Howell Test for Pairwise Comparisons of Means, and Simultaneous Test for Differences of Means was used in place of Tukey's Test.

Results

The study reports the progression in students' understanding of the concept of rates of change and functions with respect to grade level and math courses. The Boxplot in Figure 2 presents the distribution of participant's total scores. A cursory examination of the Boxplot shows that the middle fifty percent of the participants' scores on the DTRC ranged from 3 to 9.

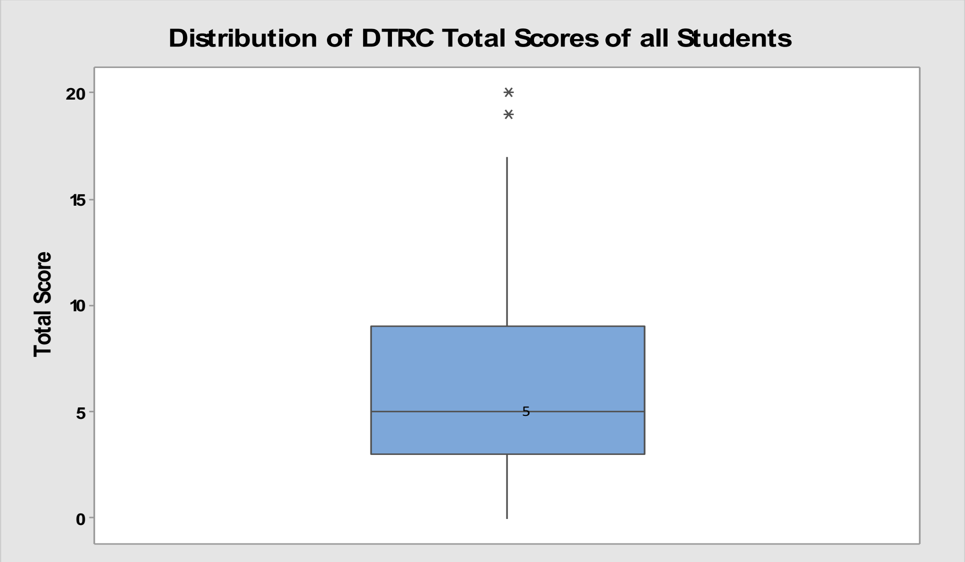

Figure 2. A Boxplot Display of the Performance of all the Participants on DTRC.

A total of six students scored 0, with only one student scoring 20. The two scores that were identified as outliers in Figure 2 are 19 (Geometry student) and 20 (AP-Calculus student).

Seventy-five percent of students (n=187) regardless of their grade level or current math course scored a 9 or less on DTRC with a mean score of 6.358 and a standard deviation of 4.133.

The Parallel Boxplots in Figure 3 present a comparative distribution of the total scores of the participants' performance on the DTRC by grade level. The information in the figure reveals a rise in scores from Grade 7 up to Grade 10. The presence of higher scores at higher grade levels may be due to increased exposure to the concepts by virtue of completion of more advanced math courses. However, Grade 10 and Grade 11 scores were similar. A high variability was seen in the score distribution of Grade 12. Students in the third quartile of Grade 12 distribution earned lower scores than the third quartile students in Grade 10 and Grade 11.

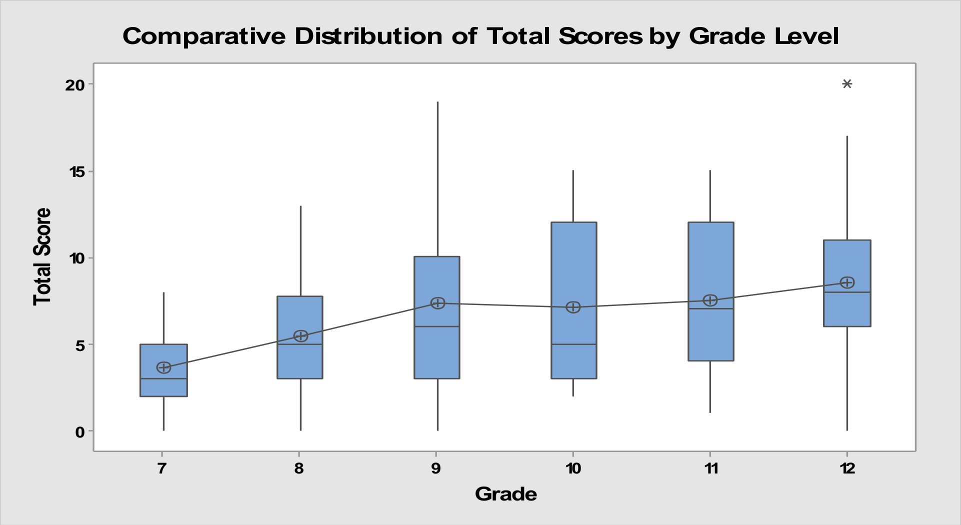

Figure 3. Parallel Boxplots for Comparative Distribution of Total Scores by Grade Level.

One-way ANOVA on Total Scores versus Grade Level, and Games-Howell Simultaneous Tests for Differences of Means showed that mean score for Grade 8, Grade 9, Grade 10, and Grade 11 were not statistically significantly different from one another. The mean score for Grade 7 were significantly lower than the mean scores of the other grade levels as expected. However, there was a significant difference between mean scores of Grade 8 and Grade 12 possibly due to an outlier score in Grade 12. In conclusion, student performance on the DTRC varied little across grade levels.

The parallel Boxplots in figure 4 show a comparative distribution of DTRC Total Scores by Math Course. The figure shows that participants in Math-7 have the lowest mean, median, and maximum score on DTRC as they were not introduced to various representations of rates of change thus far. PreAlgebra, and Math 8 groups show approximately equal mean scores, close values for median, and for maximum scores. Students in Geometry and Precalculus groups have similar performance with 75% of students scoring 10.25 or less in Geometry, and same proportion of students scoring 9.5 or less in Precalculus. Though not expected, 75% of Precalculus students (Grades 11, 12) scored 9.5 or less which is almost the same as lower fifty-percent of Algebra II (Grades 10, 11) group.

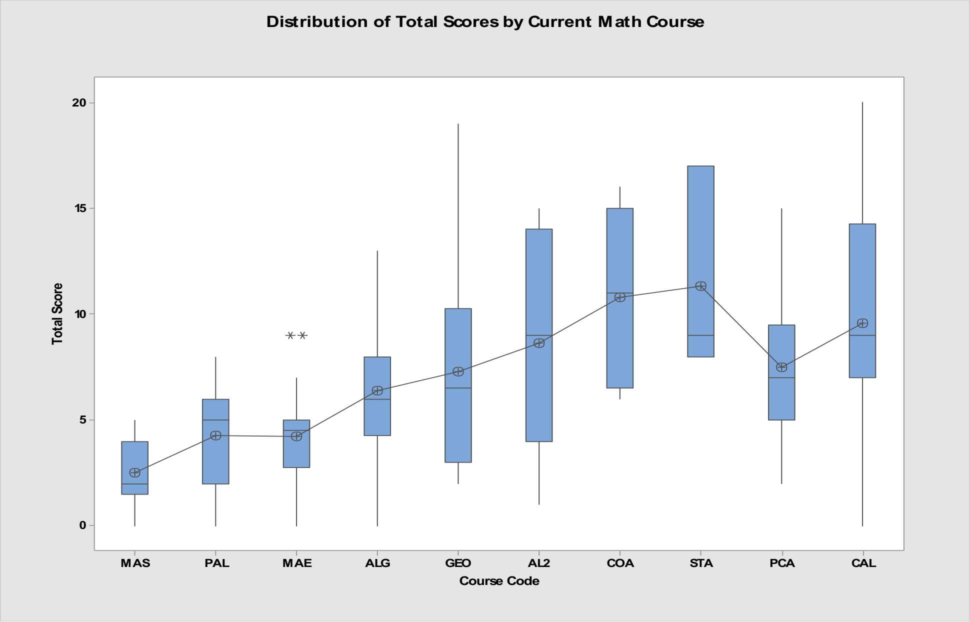

Figure 4. Parallel Box Plots of Total Scores by Math Courses.

Also, it was expected that AP-Calculus group would outperform student groups from the other math courses. Only one AP Calculus student scored a 20. Three-fourths of the AP-Calculus participants scored 14.25 or less, as did the same proportion of students in Algebra II. DTRC participants enrolled in College Algebra, and AP-Statistics scored higher than the participants from the other math courses in the middle fifty-percent.

Based on One-way ANOVA of Total Scores versus Course Code, mean scores for Algebra I, Geometry, Algebra II, Precalculus, and AP-Calculus were not statistically significantly different from one another. DTRC mean scores for each math course are much lower than expected score corresponding to that math course (Table 2). Students' understandings do not appear to consistently build in a manner that could lead students to develop a solid mathematical structure of functions at the end of Precalculus.

Scholarly Significance of the Study

The focus of the study was to identify similarities and differences of grade seven through grade twelve students' understanding of rates of change and functions while keeping in mind that the participants from the higher grades will have more mature mathematical thinking in comparison to the participants from lower grades. This study found that students' performance on the DTRC test did not show a similar progression in the internalization of the concepts of rates of change, and in formulating mathematical structure of functions as prescribed in the state curriculum standards. An implication of this study is for teachers to carefully select mathematical tasks that allow students to explore amounts of change and relationship between covarying quantities in the dynamic real-world contexts. This instructional intervention will provide the necessary experience for students to develop covariational reasoning, and to have a better understanding of functions in general.

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TINKERPLOTS AND STUDENT UNDERSTANDING OF DATA DISPLAYS

Lucas Foster, PhD.
Northeastern State University
fosterlb@nsuok.edu

Students often encounter situations where they make choices requiring a thorough understanding of mathematics and statistics. In order for these students to make informed decisions, they must be statistically literate. Teachers try numerous strategies, including the incorporation of different technology into the classroom, to help their students become mathematical and statistical thinkers. One particular technology, Tinkerplots, a dynamic statistical software program, has been used in a variety of settings to enhance the mathematics classroom. The purpose of this study was to determine whether the use of Tinkerplots in a statistics classroom affects students' understanding of graphical displays of data.

Purpose of the Study

Students are frequently exposed to real-world situations in which they are required to make decisions where a deep understanding of statistics is needed. In order to be adequately prepared to make informed decisions, students need to be able to think and reason mathematically about statistics. *Statistical literacy*, which can be defined in a multiplicity of ways, but is best defined by Gal (2002), is an important part of the educational process of students. Gal's (2002) definition of statistical literacy includes students' ability to interpret and critically evaluate statistical information and their ability to discuss or communicate reactions to such statistical information, such as the understanding of what the information means. So according to Gal's definition, for a student to be considered statistically literate, he or she must be able to understand and interpret graphical displays of data. *Tinkerplots* is a dynamic computer software program, developed in 2005 by Cliff Konold and Craig Miller, which was designed for the purpose of helping students understand and interpret statistical data. The purpose of this study is to determine if *Tinkerplots* can significantly affect student understanding and interpretation of graphical data displays.

Statistical Literacy

The importance of statistical literacy, particularly numerical data and their representations, pervades society in an array of ways--news, politics, finance, medicine, etc. Steen (1999) suggests that "[t]he age of information is an age of numbers"(p. 8). However, studies show that students in the United States finish high school without adequate numerical and statistical reasoning. Businesses lack employees having quantitative or technical skills while colleges and universities must offer a wide variety of developmental courses to offset numerical deficiencies of incoming students. "Despite years of study and experience in an environment drenched in data, many educated adults remain innumerate" (Steen, 1999, p. 9). In *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford, & Findell, 2001), it is reported that research evidence is both consistent and compelling in showing that students in the United States are weak in mathematical performance. Assessments indicate that U.S. students can adequately perform straightforward arithmetical procedures, but demonstrate only a limited understanding of mathematical concepts. Furthermore, students cannot apply mathematical skills to solve simple problems.

As it becomes clearer that students are numerically and statistically deficient, our society is increasingly dependent on citizens and workers with quantitative and numerical skills. Mullis, et.al. (2004) claimed, according to the TIMSS 2003 International Mathematics Report, that although some students were performing better in mathematics and statistics, the United States still ranks well below the leading countries. In addition, particular sectors of the general population are underrepresented by individuals who are successful in mathematics and statistics. Some people cannot participate fully in our society because they lack a basic understanding of mathematics.

Changes in most professional and educational fields require numerical and statistical literacy as necessary ingredient in all domains more than ever before (Lakoma, 2007). Steen (1990) makes the case that although numeracy (mathematical literacy) is to mathematics as literacy is to language, both literacy and numeracy are in decline in the United States. Even as it becomes more imperative that students have quantitative skills, the workforce is less mathematically or statistically literate. If individuals are not numerically literate and not able to think critically about data, then they are unable to participate in any discussion about what numbers mean (Whitin & Whitin, 2008).

Tinkerplots

Tinkerplots was developed with the goal of enhancing statistics education by enabling students to interact with data in a dynamic and fluid computer environment. Numerous research has been conducted on the effectiveness of *Tinkerplots* in a variety of settings with promising results. According to Lee and Hollebrands (2008), programs like *Tinkerplots* were created with the intent of giving users the ability to have dynamic control over data. This enables users to reorganize data to gain a different perspective and amplify the abilities of users to solve problems. When discussing the different strategies students employ when interacting with *Tinkerplots*, Fitzallen (2013) identifies three different strategies—*Snatch and Grab, Proceed and Falter, and Explore and Complete*. Fitzallen notes that students using "the *Explore and Complete* strategy make strong connections between the graphical representations and the meaning they embody" (p. 13). She further states that *Tinkerplots* is most suited to students using this strategy. In a study on the use of *Tinkerplots* as a tool to understand the relationship between data, Monteiro, Asseker, Carvalho, & Campos (2010) identify the way in which participant interpretation of data changed from a local to a global approach when the "participants engaged in the process of transform(ing) the representations using *Tinkerplots*" (p. 5). Watson and Wright (2008) report that *Tinkerplots* enhances the learning environment by saving time and adding "creativity and student ownership to the production of evidence and the creation of a final report answering the initial questions" (p.40).

In *Resampling with Tinkerplots* (2012), Watson describes how students are able to manipulate sets of data and gain intuition regarding claims made about the data without using formal hypothesis testing methods. "Students are able to gain intuition about what is required to accept or reject a claim made for a difference in two populations long before they meet formal *t*-tests at a university or senior secondary level. The concrete visual approach of *Tinkerplots* can provide the hands-on experience students need to develop such intuition" (p.36). Hall (2008) echoes this sentiment when stating that participants found the program intuitive, user-friendly, and a complement to learning. A similar experience is described by Fitzallen & Watson (2010) when they discovered that students were able to become independent users of *Tinkerplots* and shifted from using the program in a simple, procedural manner into using it in a creative, discriminating way. Students exhibited that they could transform simple data analysis skills into reasoning about representations of data. "In doing so, they created plots that made sense to them and used the plots effectively to support their thinking about the data" (p. 5).

Methods

The purpose of this study was to determine if the use of *Tinkerplots* in a college statistics classroom has a significant effect on the students' level of understanding of graphical representations of data. Pre-post treatment quantitative data were collected and analyzed to determine students' level of understanding of data displays. The instrument used to collect the data, the Statistics Assessment, was checked for both internal reliability and face validity. The Cronbach's alpha reliability coefficient for the participant Statistics Assessment scores was 0.769. According to George and Mallery (2003), this reflects an acceptable internal reliability. Items on the Statistics Assessment focused on students' ability to understand and interpret graphical displays of data. Examples of Assessment items are shown in Figures 1 and 2.

Participants in the study were selected from the general population of statistics students at a Midwestern regional university. Participants were enrolled in one of two sections of a Statistical Methods course, with one section being randomly selected as the control group and the other being selected as the experimental group. A convenience sample of 42 students participated in the study—19 in the control group and 23 in the experimental group. Both sections of the course were taught by the same instructor, in the same computer lab, on the same days of the week, and meeting for the same amount of time. The curriculum and pedagogy in both sections were the same, with the only difference being that participants in the experimental group were involved in completing *Tinkerplots* activities rather than completing supplemental in-class problems. Participants in the experimental group completed learning activities by working in groups during class time at computer stations and using *Tinkerplots*. The learning activities consisted of problem sets designed to assess students' understanding of data displays. The experimental time-frame was four weeks.

Participants in the experimental group completed *Tinkerplots* activities that were chosen specifically because of how they reflected understanding and interpretation of data displays. Participants in the control group completed supplemental activities that were part of the original course curriculum. The experimental group was given an orientation to the *Tinkerplots* program during the first class period in the computer lab prior to the first activity. At the conclusion of the orientation, participants in the experimental group were able to complete the activities using *Tinkerplots* for the entire experimental time frame.

Using the graph below, what conclusions can you draw regarding the lengths of normal fish vs. genetically engineered fish?

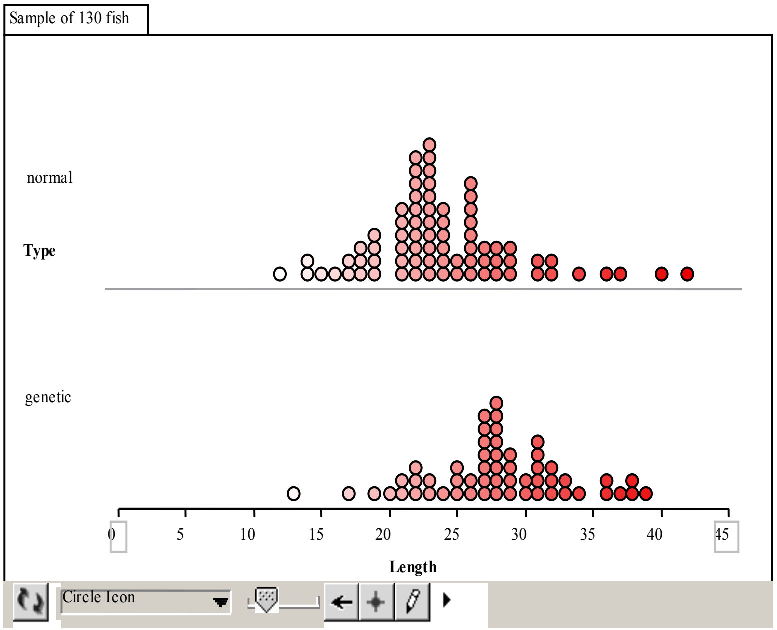

Figure 1: Statistical Assessment Example Item #1

Each subject in the study ($N = 42$) completed the Statistics Assessment during the first class meeting at the beginning of the experimental time frame. The investigator distributed and collected the Statistical Assessment instruments. Participants spent four weeks completing the *Tinkerplots* module in the statistics course. After completion of the module, subjects in both sections were again given the Statistical Assessment, which was distributed and collected again by the same investigator.

Results

The number of participants in each group ($N < 30$) necessitated the use of an independent samples Mann-Whitney U test for analysis. Using this test on the pre-treatment Statistical Assessment scores determined there was no significant difference between the experimental group and control group at the beginning of the experimental time frame. Results of the Mann-Whitney U test indicated there was no significant difference found between the two groups, $U = 157.5$, $p > 0.05$, with the sum of ranks equal to 433.5 for the experimental group and 469.5 for the control group.

After the *Tinkerplots* module, an independent samples Mann-Whitney U test was conducted on the post-treatment Statistical Assessment scores at the $\alpha = 0.05$ level to determine if there was

What inferences can you make using the following scatter plot?

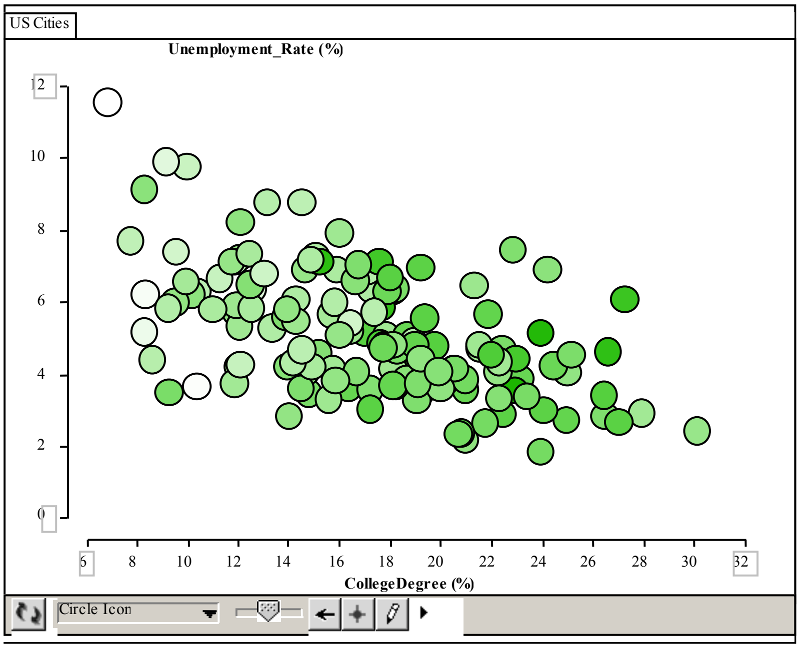

Figure 2: Statistical Assessment Example Item #2

a significant difference between the experimental and control groups. Analysis revealed no significant difference found between the two groups, $U = 218.0, p > 0.05$, with the sum of ranks equal to 495 for the experimental group and 408 for the control group.

Table 1

Descriptive and Inferential Statistics of Pre-Post Statistical Assessment Scores

Group	Range	Mean	Rank Sum	*U*	*p*
Pre-Treatment Scores					
Control	16	36.53	469.5	157.5	0.121
Experimental	08	34.17	433.5		
Post- Treatment Scores					
Control	15	38.32	408.0	218.0	0.990
Experimental	19	38.65	495.0		

Note. N = 19 for the control group; N = 23 for the experimental group

The results of the assessments analysis are presented in Table 1. It should be noted that, although the difference on post-treatment Assessment scores was not significant, the experimental group scored higher, on average, than the control group. In addition, the net difference between average assessment scores went from a -2.36 to a + 0.33 from the experimental group perspective.

Discussion

Data analysis indicates that the influence of *Tinkerplots* on students' understanding of graphical representations is minimal. According to the analysis of the quantitative data from the Statistical Assessments, there was no significant difference found between the control group and the experimental group which suggests that *Tinkerplots* did not significantly affect the ability of students to understand or interpret data displays. This is somewhat unexpected when compared to results reported by Fitzallen (2013), Watson & Wright (2008), and others. However, when one considers the mean scores on the Assessments from both groups and the change in those scores from pre to post-treatment, it is hard to ignore that the experimental group went from underperforming the control group to outperforming them, albeit slightly. This could indicate that *Tinkerplots* has a greater influence on students who struggle with interpreting data displays, and helps them understand graphs as well as students who do not struggle. However, this is not a claim, simply an idea for further research. Finally, participants in this study only used *Tinkerplots* for four weeks. It would be beneficial to investigate if a longer *Tinkerplots* module, say a semester, would vary the results. A prolonged experience with *Tinkerplots* would alleviate stress or trepidation when the program is introduced. In any case, it would be worthwhile to increase the body of research on the overall effectiveness of *Tinkerplots*.

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CALCULUS STUDENT UNDERSTANDING OF CONTINUITY AND RELATED TOPICS

Jayleen Wangle
State University of New York at Oneonta
Jayleen.Wangle@oneonta.edu

Continuity is a central yet subtle concept in Calculus I. Yet very few students seem to grasp the nature of continuity. The purpose of this study was to investigate participant displayed depth of understanding of function, limit, and continuity in terms of the constructs described by Dubinsky's (1991) Action-Process-Object-Schema theory. This paper reports on interview results of two participants. A prominent finding was that the participant who was more successful at solving problems concerning continuity displayed an Inter-level of development of schema of function and limit and the participant who less successful demonstrated that he was at the Intra-levels.

Apostol (1967) describes continuity as "one of the most important and also one of the most fascinating ideas in all of mathematics" (p. 126). Many of the theorems presented in calculus begin with the hypothesis of continuity. Yet many textbooks contain only one section that explicitly addresses the concept of continuity (Hass, Weir, & Thomas, 2008; Stein, 1987; Stewart, 2014). This treatment of continuity may lead students to believe that continuity is not important. Furthermore, since the notion of limit is embedded within the definition of continuity, student comprehension of limit strongly influences their understanding of continuity.

Many textbook authors discuss continuity in terms of the graphical representation of function. Intuitively a function is said to be continuous if one can sketch its graph without lifting one's pencil (Hass et al., 2008; Stein, 1987; Stewart, 2014; Taalman & Kohn, 2014). Tall (1991) claims that describing continuity in these informal terms is misleading. Furthemore, Tall (1991) argues that this intuitive explanation lays the groundwork for connectedness and not continuity. Millspaugh (2006) argues that presenting continuity in this way confuses students concerning the role of limit.

Theoretical Framework

Action Process Object Schema (APOS) theory is an extension of Piaget's reflective abstraction developed as a framework for describing student learning of advanced mathematical topics like calculus (Dubinsky, 1991). The theory is based on the premise that an individual attempts to solve mathematical problems by mentally constructing *actions*, *processes*, and *objects*, and then organizing them into *schemas* (Arnon, et al., 2014; Asiala et al., 1996;

Breidenbach Dubinsky, Hawks, & Nichols., 1992; Cottrill et al., 1996, Dubinsky, 1991; Dubinsky, & McDonald, 2001). APOS theory provides researchers with tools to describe demonstrated levels of student understanding. The purpose of this study is to use the constructs of APOS Theory to investigate demonstrated levels of student understanding of continuity, and their relationship between demonstrated levels of student comprehension of function and limit.

An *action* is a step-by-step procedure that transforms an object. The cues to perform these steps are viewed primarily external to the student. If one demonstrates a view of function limited to action, one will be able to fill out a function table, but one will view each calculation of range values in isolation from the rest (Briedenbach et al., 1992). An individual restricted to an action conception of limit will evaluate *f* only at *a* or at a few values near *a* when computing $\lim_{x \to a} f(x)$ (Cottrill et al., 1996).

A *process* results from an individual repeatedly performing an action, reflecting on that action, and then interiorizing that action. An individual no longer needs external stimuli to perform the same action. Furthermore, an individual demonstrates a process level of understanding by describing, reversing or composing that action without explicitly carrying out the steps. A student that appears to be able to view a function as a *process* will notice that each range value of the function is obtained by applying the same computations to its corresponding domain value (Breidenbach et al.,1992). According to Cottrill et al. (1996), one displays a process view of limit by coordinating the domain and range processes in order to calculate $\lim_{x \to a} f(x)$. The domain process results from interiorizing the action of taking domain values that successively approach *a*. The range process involves evaluating *f* at the points from the first process and noticing if they appear to be close to a certain number *L* when the domain values are sufficiently close to *a* (Cottrill et al., 1996). So a process conception of limit depends on a process view of function.

An *object* results from an individual reflecting on a procedure applied to a particular process and becoming conscious of that process as a totality. One displays an object view of function by showing a facility when working with transformations of functions (Arnon et al., 2014). An individual will show an object view of limit by being able to find limits of combinations of functions (Cottrill et al., 1996).

An individual's *schema* of a particular mathematical concept consists of actions, processes, objects, and other schemas linked in a coherent manner. It is a framework from which an individual solves problems that pertain to that concept (Asiala et al., 1997) and develops in three stages: Intra-, Inter-, and Trans- (Baker, Cooley, & Trigueros, 2000; Clark et al., 1997; Cooley, Trigueros, & Baker, 2007; Dubinsky & McDonald, 2001).

The Intra- stage is characterized by a focus on individual cognitive constructs in isolation from all others. One who appears to show an Intra-function stage of progress will tend to focus on one representation of function within a specific context and the actions that one can do with that (Dubinsky & McDonald, 2001). While Cottrill et al. (1996) do not address the stages of the limit schema as a triad, I put forth that since a process conception of limit requires a coordination of processes, one who is at the Intra-limit stage of development will be restricted to an action view of limit.

The Inter- stage is marked by the ability to coordinate relationships between the Processes and Objects of which the schema is composed and is referred to as a *pre-schema* stage of development (Arnon, 2014; Clark et al., 1997). In particular, one who demonstrates an Inter-function stage of growth will be able to successfully coordinate the algebraic and graphical representations of the same function. Furthermore, such an individual will demonstrate an ability to understand that the different operations used to combine two functions are examples of the same action, namely transformation of functions (McDonald & Dubinsky, 2001). So one at this stage will be able to think of a function as an object. An individual who appears to have reached the Inter-limit stage of progress will be able to think of a limit as a coordination of processes as described above and a finite object; it is not necessary to have an object conception of limit. In addition an individual will be able to calculate the limit correctly for the same function given in algebraic and graphical form.

One is said to have reached the Trans- stage when one's cognitive structures are connected in a coherent manner (Baker, Cooley, & Trigueros, 2000; Clark et al., 1997; Cooley, Trigueros, & Baker, 2007; Dubinsky & McDonald, 2001). At the Trans-function stage, an individual will recognize that all functions have a domain and range and an operation or series of operations that that transform each domain element into a range element (Arnon et al., 2014; Dubinsky & McDonald, 2001). At the Trans-limit stage one will show the understanding that $\lim_{x\to a} f(x) = L$ if and only if for any value x arbitrarily close to a, $f(x)$ is sufficiently close to L. One at the Trans-limit stage will be able to coordinate the above description with an object conception of function to find the limit of any combination of functions.

Even though the above discussion makes it appear as though the constructs of APOS Theory are developed in a hierarchical fashion, the reality is more complicated than that. Consider the mathematical concept of function. Initially an individual is exposed to certain kinds of functions. Later the individual may develop a process conception of function with regards to those types of functions, but the individual will think of a more sophisticated function as an action when first working with it. So one may demonstrate a different conception of function when working with various functions depending upon one's experience with each function (Arnon et al., 2014; Dubinsky & McDonald, 2001).

When comparing two participants who up to a certain point may be successful but with more sophisticated problems one may show more success than the other, the theory is useful in describing the mental constructs that the more successful student has made that the less successful student has not (Arnon et al., 2014; Dubinsky & McDonald, 2001).

Method

Participants for this study were drawn from two Calculus I classes at a mid-sized university in the Midwest. To control for differences in instruction, I developed a curriculum for teaching continuity for both instructors to use. The curriculum was traditional and did not use connectedness of a graph to introduce continuity. I attended their classes to confirm that each of them was teaching from my curriculum. I distributed written instruments: one each for function, limit, and continuity to the students in each of these classes. Eight interviewees were chosen based on their written responses and instructor input so that the sample contained participants of varying levels of ability. The interview tasks consisted of problems to solve pertaining to function, limit, and continuity. Each participant was interviewed once for a period of about 50 minutes. Interviews were video-taped and then transcribed. The results of the interviews were used to answer the following research questions.

1. What are the ways that participants think about continuity?
2. What is the relationship between demonstration of Intra-, Inter-, and Trans- levels of development of participant function schema and how well participants solve non-routine problems concerning continuity?

3. What is the relationship between demonstration of Intra-, Inter-, and Trans- levels of development of participant limit schema and how well participants solve non-routine problems concerning continuity?

Working Definitions

For the purposes of this study, the following definition of continuity at a point will be used: Let a function *f* be a defined on an open interval containing the real number a. Then *f* is continuous at a if $\lim_{x\to a} f(x) = f(a)$. It is also necessary to clarify the nature of the questions employed. A non-routine problem is one that is not assigned for homework and does not appear on any exam (Selden et al., 1999).

Findings

Even though eight students were interviewed, this paper reports sample responses of two participants, Harold and Steven. Of the eight students interviewed, Harold worked the most problems successfully. The sample of responses reported shows that Harold demonstrated he was at the Inter-function stage on the items with functions with which he had more familiarity. Steven demonstrated he was at the Intra-function stage.

In the following instances, Harold successfully determined that the function is not continuous on its domain. The first instance regards the function $f(x) = \begin{cases} \frac{1}{8}(x+1)^2 + 3, & x < -1 \\ -\frac{1}{8}(x+1)^2 + 1, & x \geq -1 \end{cases}$ and the second is the graphical form of the same function. He answered the question regarding the algebraic form first without having seen the graphical form. In these instances Harold demonstrated an Inter-function stage: he was able to view a function from an object perspective and he successfully coordinated the graphical and algebraic forms of the function. In the excerpt below I had asked him to clarify how he reached his conclusion regarding the domain of the above function. (Note that initially he gave an incorrect response but then corrected himself.)

> Harold: Cause uh the limit from the right is not equal to the limit from the left. So there is a gap in between the … (He demonstrates with his hands). No the domain is actually from minus infinity to infinity. It is not continuous at negative one, because the value of the limit as *x* goes to minus one from the right is not equal to the limit as *x* goes to negative one from the left.

Later on using limits he explained why the graphical form of the same function was not continuous at $x = -1$. When I asked him if there was a relationship between the two functions, he gave me the following response.

Harold: …there is a shift uh shift to the left there is like a shift upward three … and it is negative so it will be an upside down parabola. This is the half of it. From *x* bigger than minus one and there is a shift up one.

In this instance Harold demonstrated that he understood the relationship between the graphical and algebraic representation of the same function. He showed that he grasped that when the left hand and right hand limits are not equal but finite, then there is necessarily a gap in the graph at that point. He also displayed an object conception of function by recognizing each of the parts that make up the piecewise defined function as transformations of the function $f(x) = x^2$. Harold gave a similar response when I asked him about other functions that were familiar to him. In these instances, Harold was exhibiting he was at the Inter-function stage.

In the above instance, Harold was able to calculate the necessary limits with ease. He also had no difficulty with realizing that these limits were finite values. In other instances, when calculating limits for functions with which he was familiar he gave me a similar explanation. So in these instances, he was showing that he was at the Inter-limit stage.

Steven on the other hand, had difficulty doing the tasks I asked of him without graphing the functions on a graphing calculator. He had a preference for the graphical form of the function. He also reasoned inconsistently with the algebraic and graphical forms of the function as shown in the discussion below pertaining to the same function that was referred to above. When I asked Steven if the above function was continuous at negative one, he said yes, explaining that the function was defined at negative one. When I asked him if the function was continuous on its domain, he said yes because, “there’s no cusp, no crease, no open circle. There’s no value that’s not evaluated at any given *x*.”

Steven did not know how to graph piecewise defined functions on his graphing calculator, but from his description, it appears that he was imagining what the graph looks like. I had given him paper to work out problems, but he did not sketch any graphs at this point in the interview. In the next portion of the interview, I handed him a card with the graphical form of the same function.

I: Is that function continuous at *x* equals negative one?

Steven: No.

I: Why not?

Steven: There's a gap between the two functions.

I: Can you tell me what the limit is as *x* approaches negative one?

Steven: One.

I: And how did you reach that conclusion?

Steven: Cause there's um there's a value of *y* evaluated at *x*.

…

I: Is that function continuous on its domain?

Steven: No it's not.

I: And why is that?

Steven: Cause there's a gap between the two functions of the piecewise.

After I asked him if there was a relationship between the two forms of the function, he recognized that they were different representations of the same function, but he was unable to give me an adequate explanation for why. The excerpts show, however, that he failed to recognize that he had been reasoning in an inconsistent manner. His preference for the graphical form of functions, his reasoning about continuity in terms of the graphical form, and his difficulty coordinating the algebraic and graphical forms of the function give evidence he was at the Intra-function stage. Furthermore, since he evaluated the limit by only evaluating the function at the point, he was showing that he was at the Intra-limit stage.

Discussion and Conclusion

It is important to note that the small number of interviewees limits the generalizability of this study. A prominent finding of this study is that when a participant solved a continuity problem correctly, he demonstrated Inter-function and Inter-limit stages of schema. Both of these stages require an object conception of function. Before one is able is to view a function as an object, one must be able to view a function as a process. If one is only able to do actions with functions, then one will have difficulty developing a sound understanding of limit and continuity.

Since grasping both of these ideas is central to success in calculus, this study points to the need for students in courses prerequisite to calculus to be given opportunities to develop at least a process conception of function. Students need to be able to reason consistently with algebraic and graphical forms of functions. This requires that students not only be exposed to both, but have the opportunity to see the relationship between the two forms and that they complement one another to give a deeper understanding of the function in question.

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UNDERGRADUATES REPRESENTING AND CONNECTING IN MATHEMATICAL PROBLEM SOLVING

Kathryn Rhoads
The University of Texas at Arlington
kerhoads@uta.edu

James A. Mendoza Epperson
The University of Texas at Arlington
epperson@uta.edu

R. Cavender Campell
The University of Texas at Arlington
robertcc@uta.edu

This study explores how entry-level undergraduate students engage in representing and connecting in the process of mathematical problem solving (MPS). Data comes from (a) Likert survey items from 254 College Algebra students and 405 Calculus I students and (b) individual, task-based interviews with 26 of these students at a large, urban university in the southwest United States. Students used representing/ connecting in MPS in three ways during interviews, but for the majority of solution paths, students showed no evidence of representing/ connecting. In addition, preliminary survey data indicates that students' representing/ connecting does not increase by participating in College Algebra.

The majority of undergraduates who intend to major in science, technology, engineering, or mathematics (STEM) do not complete a degree in a STEM field, and difficulty in mathematics is one contributing factor (President's Council of Advisors on Science and Technology, 2012). Some research indicates that undergraduates may struggle with mathematical problem solving (MPS) in particular (e.g., Schoenfeld, 1992). MPS is fundamental to mathematics, yet there is limited research on how undergraduates engage in MPS (Schoenfeld, 2013).

In the Mathematical Problem Solving Item Development Project, we aim to develop efficiently-scored survey items assessing undergraduate students' MPS in five key domains, including *representing and connecting*. In this study, we explore the following: (a) How do undergraduates engage in representing/ connecting in MPS? (b) How do students' survey scores in representing/ connecting change through participation in undergraduate mathematics courses?

Theoretical Framework

Lester (2013) asserted that "a problem is a task for which an individual does not know (immediately) what to do to get an answer" (p. 247). With this definition of *problem* in mind, we considered characterizations of what constitutes MPS. Campbell (2014) analyzed more than 20 research articles in MPS and classified implied or explicitly-stated definitions of MPS, identifying five key characteristics: emphasis on sense-making, connecting mathematical ideas in various ways including via multiple representations, reviewing or reflection, justification, and underlying challenge. For example, Schoenfeld (1988) explicitly referred to sense-making and taking apart a problem to seek understanding, and Kieran's (2007) references to combining previously-learned techniques imply connecting prior knowledge to problem situations. Jonassen (1997) also highlighted connections, arguing "problem solving, as an activity, is more complex than the sum of its component parts" (p. 65), and Garofalo and Lester (1985) included monitoring and checking as important aspects of MPS. Researchers including Jonassen (1997) discussed that justifying or defending generated problem solutions promotes deep learning.

Epperson, Rhoads, and Campbell (2016) refined definitions of Campbell's domains of sense-making, representing/connecting, reviewing, justifying, and challenge. Representing/connecting was defined as "Bridging the problem to another idea, related mathematical approaches, or representations. Reformulating the problem by using a different representation, or connecting the problem to seemingly disjoint prior knowledge." (p. 2; see Epperson et al., 2016 for definitions).

Methodology

Participants

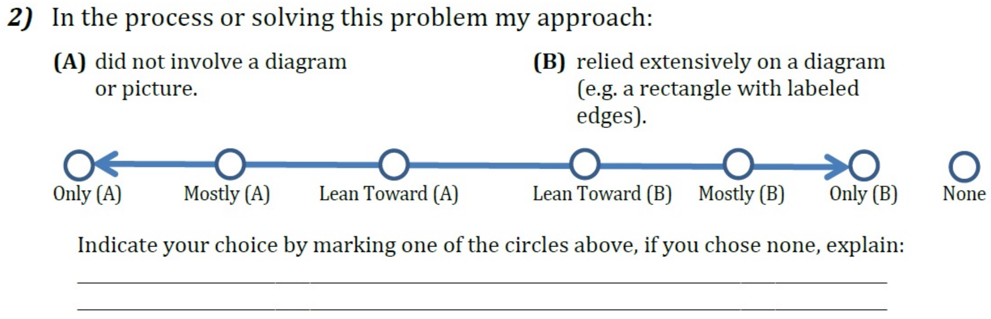
Participants were 254 College Algebra students (108 Fall 2015 and 146 Spring 2016) and 405 Calculus I students (Fall 2015) at a large, urban university in the southwest United States. Both courses are designed for students intending to major in STEM. College Algebra participants came from nine sections with five instructors in Fall 2015 and four sections with three instructors in Spring 2016. Calculus participants came from eleven sections with eight instructors.

MPS Survey

To assess the five domains of MPS, we created open-ended mathematics problems and associated Likert-style survey items and administered a pre- and posttest for the semester course. In a single test, each student solved five mathematics problems, which were designed to be appropriately challenging yet not require knowledge beyond secondary-school algebra. Participants were not exposed to the particular problems before the pretest. After solving, students completed five to seven Likert items for each problem (25 – 35 items total), and each item assessed one MPS domain (see Figure 1). Each item suggested two approaches (one high use of the domain and one low use of the domain), and students rated their approach.

Interviews

We conducted individual interviews with 19 College Algebra students (from seven instructors) and seven Calculus students (from five instructors) between the pre and posttests. In the one-hour video-recorded interviews, students were asked to explain their thinking behind their written work and survey items for (a) at least three problems that were solved on the pretest and (b) one new problem and associated items.



2) In the process or solving this problem my approach:

(A) did not involve a diagram or picture.

(B) relied extensively on a diagram (e.g. a rectangle with labeled edges).

○ Only (A) ○ Mostly (A) ○ Lean Toward (A) ○ Lean Toward (B) ○ Mostly (B) ○ Only (B) ○ None

Indicate your choice by marking one of the circles above, if you chose none, explain:

Figure 1. A representing/ connecting survey item for the Ken's Garden problem.

Data Analysis

We scored survey responses 1 (low) – 6 (high) according to the student's choice and recorded domain averages for each student. For interviews, MPS domains were used as a coding framework (e.g., Miles & Huberman, 1994). Each researcher independently coded 3-5 interviews. We then compared and revised coding before analyzing the remaining interviews.

For the domain of representing/ connecting, survey items and the coding scheme gave preference to visual representations (written or mental) and mathematical connections because we hypothesize these are non-traditional MPS approaches for entry-level undergraduates. As such, our interpretation does not capture all types of representing/ connecting.

Results

How Undergraduates Engage in Representing and Connecting

Summary. Undergraduates used representing and connecting (R/C) in three main ways during MPS in interviews: (a) visualizing or sketching graphs, (b) drawing diagrams, and (c) making mathematical connections to other problems or ideas. Despite a few rich illustrations of R/C, students showed no evidence of R/C (written or verbal) for the majority of solution paths, and in approximately half of the instances of R/C, problem statements may have prompted the use of R/C. Below we present counts from interviews followed by description and examples.

Table 1 shows the number of students who verbalized or explained a representation or connection during MPS. Table 2 indicates the number of students who discussed each type of R/C. Some problems referred specifically to graphs or alluded to geometric diagrams (e.g., a rectangular garden). In Table 3, the number of solution paths in which students discussed R/C is reported according to problem type. The total number of solution paths consists of all solutions that students discussed during the interviews.

Table 1

Number of students discussing representing/ connecting in interviews

	College Algebra	Calculus	Total
Discussed representing/ connecting	12	5	17
Did not verbalize representing/ connecting	7	2	9
Total	19	7	26

Table 2

Number of students discussing each type of representing/ connecting in interviews

	College Algebra (n=12)	Calculus (n=5)	Total (n=17)
Visualizing/ sketching graphs	8	2	10
Drawing diagram	8	4	12
Connecting	2	1	3

Table 3

Number of solution paths in which students discussed representing/ connecting in interviews

	Graph/ Diagram in problem statement	Graph/ Diagram not in problem statement	Total
Visualizing/ sketching graphs	6	6	12
Drawing diagram	9	8	17
No graphs or diagrams discussed	10	73	83
Total	25	87	112

In some instances, a student had drawn a diagram or graph, but did not discuss it in the interview. Table 4 shows the number of solution paths in which there was *any* evidence of R/C, separated into verbal and written evidence.

Table 4

Number of solution paths in interviews with any evidence of graphs or diagrams

	Graph/ Diagram in problem statement	Graph/ Diagram not in problem statement	Total
Described visualizing/ sketching graphs	6	6	12
Described drawing diagram	9	8	17
Graphs sketched, not described	0	1	1
Diagrams sketched, not described	4	8	12
No graphs or diagrams	6	64	70
Total	25	87	112

Description. Eight college algebra students and two calculus students described how they visualized or sketched graphs as part of their problem-solving process in 12 total instances. In six of the 12 instances, the problem specifically referred to graphs (see Tables 2 & 3). For example, consider the Building Functions problem:

> The graph of the function g contains the points, (3,11), (-1,3), (5,15), (-4,-3), and (-7,-9). Which of a) $f(x) = 2^x$, b) $f(x) = x^2$, c) $f(x) = \sqrt{x}$, d)$f(x) = x$ can be used to build g by replacing $f(x)$ with $af(x)$ or $af(x) + k$, where $a \neq 0$ and $k > 0$? Explain your reasoning.

This problem refers to the graph of the function g, which may prompt students to use graphs in solving the problem (although visual approaches are not required for a valid solution). Participant 1 (P1) used mostly visual approaches to solve the Building Functions problem. P1 plotted the points on a coordinate plane and then compared the visual pattern of the five points to mental images of the graphs of the functions a) – d). She explained:

> I had a rough visual of what the graph would be. … If you drew these [graphs of functions a) – d)] out, they would be very, very different shapes. … A parent graph is a base and is what you have to work with. There can be general changes, but it will not fundamentally change the base graph into something else.

In six of the 12 instances in which students described graphs, the problem statement did not refer to graphs. For example, consider the Fun Golf problem, which does not refer to graphs:

> Fun Golf, a local mini-golf course, charges \$5 per person to play one round of mini-golf. At this price, Fun Golf sells 120 rounds per week on average. After studying the relevant information, the manager says for each \$1 increase in price, five fewer rounds will be purchased each week. To maximize revenue, how much should Fun Golf charge for one round?

In solving Fun Golf, P2 first wrote an equation for the revenue in terms of the increase in ticket prices. Recognizing that the graph of this equation is a parabola, P2 solved the problem by reasoning about the parabola, saying, "To maximize revenue, we just need to find the vertex of the parabola." After solving the problem correctly, he reflected on his approach: "Before even considering the function of revenue, I understood that we're dealing with a parabola with a negative slope [likely referring to the leading coefficient of the equation]. And so, it's very clear that your vertex is … your highest revenue."

Eight college algebra students and four calculus students discussed diagrams as part of their problem-solving process in 17 total instances (see Tables 2 & 3). In nine of the 17 instances, students were solving a problem that alludes to a rectangular garden, the Ken's Garden problem:

> Ken's existing garden is 17 feet long and 12 feet wide. He wants to reduce the length and increase the width by the same amount. If he wants his new garden to be approximately half

the size of the current garden, what dimensions are appropriate for Ken's new garden?

All students who used a diagram in solving Ken's Garden drew a rectangle and used the rectangle to reason about the changes in length and width of the garden. In addition, there were eight instances in which students discussed diagrams when the problem statement did not allude to a diagram. For example, when P3 was solving the Fun Golf problem, she was thinking of the revenue as a product of ticket price and rounds purchased, so she drew a rectangle with the ticket price as the length, the rounds purchased as the width, and the revenue as the area. She then considered how the area of the rectangle could be maximized.

In solving problems, two college algebra students and one calculus student made mathematical connections to other ideas with which they were familiar (see Table 2). For example, in the Air Travel problem, an airplane is flying from Boston to Los Angeles, and the plane's distance from Los Angeles is given as a function of time the plane has been flying. In solving Air Travel, P4 described how the function was connected to the idea of displacement, which helped her to reason about the problem.

Students described or sketched graphs and diagrams to solve problems for 42 solution paths, and 19 of these instances were in response to problems that referred to a graph or diagram. Further, there were 70 solutions paths for which there was no written or verbal evidence that students used graphs or diagrams to solve the problem. For the 87 problem statements that did not refer to graphs or diagrams, 64 corresponding solution paths had no evidence of graphs or diagrams. Notably, for the 25 problems that did refer to graphs or diagrams, there were 6 solution paths for which students showed no evidence of graphs or diagrams (see Table 4).

Interviews further indicated that some students used graphs and diagrams only in cases where they were having difficulty solving the problem using equations or formulas. Three students specifically described such an approach. For example, P3 explained why he drew a diagram to better understand the problem: "because this is something I didn't have the practice to be able to do properly. And other [problems], it was easier for me to think, 'Okay, this is what I need. … Put this in, put this in there, plug this in, plug this in.'" Similarly, P5 said if he was solving a problem where he wrote "some kind of a function or something and it still doesn't feel right, I'll try to draw a graph or something. But usually that's forestalling the inevitable of settling on an answer I don't like anyway." That is, P5 seemed reluctant to draw a graph, and only claimed to do so when symbolic approaches were unsuccessful.

MPSI Items Measuring Representing/ Connecting

Data from the larger sample came from the survey items measuring R/C.

Table 5

Average scores for Likert items assessing representing/ connecting

	Pretest	Posttest
Fall 2015 College Algebra	3.4496 (n = 108)	3.1265 (n = 49)
Spring 2016 College Algebra	3.4346 (n = 133)	3.4955 (n = 28)
Fall 2015 Calculus	3.1816 (n = 405)	3.4919 (n = 165)

Average scores for R/C are in Table 5. R/C was rated on a scale from 1 – 6, so a middle score would be 3.5. The data in Table 5 indicates that most groups averaged very close to the middle of the scale, suggesting that strong use of R/C (score 4-6) is balanced with weak use (score 1-3). The Fall 2015 data suggested that students may use R/C more at the beginning of College Algebra than the end, but data suggests the opposite may be true for students in Calculus (see also Campbell, 2016). This conjecture is supported by data from the 12 College Algebra students who completed both the pre- and posttests in Fall 2015. For these students, the overall average in R/C decreased 0.4858, the greatest decrease of any of the domains measured, and only two of the 12 students had individual averages in R/C that increased from pre to posttest.

Spring 2016 posttest results do not indicate a significant increase in R/C scores, although these results should be interpreted with caution due to low participation.

Discussion

Research literature indicates that representing and connecting are important aspects of MPS, and this study's interviews offered some rich examples of how undergraduates can use graphs, diagrams, and mathematical connections in MPS. However, for the majority of solution paths in interviews, students showed no evidence of R/C, and some students indicated they were using graphs or diagrams reluctantly rather than as a preferred strategy in MPS. At the same time, quantitative data from surveys did not indicate that students' R/C scores improved with their participation in College Algebra—a course in which several topics are suited for R/C. Why might students have limited engagement in R/C? We do not yet have data to answer this question, but we offer several points for consideration. We wonder if undergraduate mathematics curriculum and/ or instruction may be influencing students' use of R/C. For example, what types of MPS experiences do students have in these courses, and how is MPS addressed? To begin to explore some of these questions, we are currently analyzing data on the College Algebra curriculum at the participating university. We have also revised Likert items and are collecting additional survey and interview data to further supplement these preliminary findings.

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THE HIDDEN CURRICULUM IN HIGHER EDUCATION MATHEMATICS MODELING TEXTBOOKS

Marnie Phipps
University of North Georgia
marnie.phipps@ung.edu

Patty Wagner
University of North Georgia
patty.wagner@ung.edu

The purpose of this study was to explore the hidden curriculum embedded in textbooks selected for tertiary mathematics modeling courses. These courses are typically populated by non-STEM majors who have experienced difficulty learning mathematics. The modeling course is often the last opportunity to affect change in how these students view the subject and their own capabilities. We investigated a randomly generated sample of textbooks adopted by a stratified sample of higher education institutions in one U.S. state. We found abstracted generalization of mathematics to be more important than contextual problem solving. This result is problematic for instructional reform efforts.

Tertiary-level mathematics modeling courses focus on solving real world problems by modeling them mathematically. In traditional settings, the relevant mathematics is “taught” by an authoritative other and then applied to real world problems. In this setting, the goal is to teach mathematics so that it can be applied. Newer understandings about how students learn has led to a focus on using real world contexts as a vehicle for making sense of the mathematics. This approach allows students to appreciate the usefulness of mathematics, capitalizes on their lived experiences to facilitate connected mathematical understandings, and optimizes mathematical learning (National Council of Teachers of Mathematics [NCTM], 2014a).

In our experience, the students enrolled in college-level mathematics modeling courses are typically non-STEM majors fulfilling the minimal number of mathematics courses for degree completion in higher education. Many of these students experienced difficulty learning mathematics in the K-12 school setting and their affinity for the subject has been severely diminished. We have noted that faculty, adjunct instructors, or graduate students may instruct mathematics modeling courses. Indeed, a single institution may have several types of instructors assigned to the multiple sections necessary to meet student demand.

Instructors of mathematics modeling courses are faced with meeting the institutional expectations for such a course. A departmental syllabus may outline course objectives for the course but the textbook itself often conveys larger messages about what content should be learned and how it should be taught (Fan, Zhu, & Miao, 2013; Mesa & Griffiths, 2012). These messages are implicitly conveyed to both instructor and students and can be said to represent a hidden curriculum (Giroux & Penna, 1979).

Because the mathematics modeling course is often the first and last mathematics course for many college students, we argue it represents a final opportunity to affect change in how students view the subject of mathematics and their own capabilities. Therefore, it is important to question what implicit and explicit messages are portrayed by mathematics modeling textbooks. In order to explore the hidden curriculum in higher education mathematics textbooks, we sought to answer the following research question: What implicit messages about mathematics teaching and learning are conveyed by textbooks adopted for use by higher education mathematics departments for introductory mathematics modeling courses in one U. S. state?

Literature Review

Herbel-Eisenmann and Wagner (2007) offered a framework for analyzing mathematics textbooks' positioning of students in relation to mathematics, their peers, their teacher, other people, and their own experiences. In doing so, they assigned agency to the text; that is, they treated inanimate textbooks as an entity capable of conveying both explicit and implicit messages to the reader. Furthermore, Dietiker (2013) argued that mathematics textbooks can be interpreted as narrative, where the text presents mathematics in a chronological order that is purposeful and influential.

For this study, we similarly viewed text as agent capable of positioning students, but extended this assumption to a textbook's ability to *frame* students in the eyes of instructors. In this way, the textbook similarly conveys explicit and implicit messages about what students should learn and how they can best learn it. We also acknowledged that the narrative order of presentation affects the reader's experience, but extended this to an assumption that narrative order conveys messages about student capabilities, as well as how they learn.

We wanted our examination of mathematics modeling textbooks to be evaluative as well as descriptive. This required us to identify a framework that defined effective teaching and learning, in addition to providing a structure for analyzing textbooks. In particular, we wanted to compare the textbook's implicit messages about teaching and learning with accepted understandings of teaching and learning in the mathematics education community. We therefore decided to position our analysis around NCTM's (2014a) eight elements of effective teaching and learning: 1) establish mathematical goals to focus learning; 2) implement tasks that promote reasoning and problem solving; 3) use and connect mathematical representations; 4) facilitate meaningful mathematical discourse; 5) pose purposeful questions; 6) build procedural fluency from conceptual understanding; 7) support productive struggle in learning mathematics; and 8) elicit and use evidence of student thinking. Other literature, as described in our subsequent sections, in combination with NCTM's descriptions of the components of effective teaching and learning formed the theoretical groundwork for our study and subsequent analysis. In the next section, we describe more specifically our methods and analysis.

Methods

Our goal was to conduct analyses on a representative sample of textbooks currently in use at a variety of institutions. To achieve this goal, we organized higher education institutions into stratified Carnegie classifications and randomly selected one institution from each classification within a single U.S. state. For each institution in our sample, we identified the mathematics course that most closely matched a typical mathematics modeling course. These courses focused on using mathematics to model real world problems. We identified the textbooks that were required by the instructors who taught these courses. One textbook was exclusively used by two separate institutions. The instructors of a third institution differed on their required textbook, with most instructors requiring one textbook and a couple of instructors requiring another. We included both textbooks in our evaluation. We therefore had four textbooks comprising our sample, hereon referred to as textbooks A, B, C and D. We have chosen not to reveal the titles and authors of these textbooks because our goal was to synthesize results to describe current national trends rather than to highlight the strengths and weaknesses of particular textbooks.

Collectively, we dissected 708 examples and exercises spanning nine sections contained within 145 pages between the four textbooks. For triangulation purposes, we selected linear modeling (and/or functions) primarily because each text contained material on this topic and unlike other topics we view linear growth as common to mathematical modeling. We each reviewed three textbooks, employing comparative analysis to verify findings. Our results are provided in the order of NCTM's (2014a) eight elements of effective teaching and learning.

Findings

To what extent do texts establish mathematical goals to focus learning? NCTM (2014a) described the establishment of goals as crucial to an ability to meet other elements of effective teaching and learning. Beyond simply stating goals and objectives, NCTM asserted:

> Both teachers and students need to be able to answer crucial questions: 1) what mathematics is being learned? 2) why is it important? 3) how does it relate to what has already been learned? 4) where are these mathematical ideas going? (p. 13)

Without exception, the textbooks stated objectives at the beginning of the chapter or section and demonstrated importance though italicizing, bolding, highlighting and boxing content. These emphases convey that the important mathematics includes solving linear equations and mathematical vocabulary and properties. Only textbook B addressed the importance of the mathematics by arguing that comprehension of linear growth is a useful skill for understanding our world. In comparison, textbook C offered that learning mathematics could potentially be of value in other subjects, or, barring that, serves as "brain exercise." None of the textbooks authentically related linear functions to previous material within the text, other than a few cursory references to a previous example or subsequent chapter. We noticed the how, why and where questions from above were primarily left to the instructor and the students to answer.

To what extent do texts promote reasoning and problem solving? Smith and Stein (1998) developed a framework for classifying mathematical tasks by focusing on the task's *cognitive demand*. Tasks requiring *memorization* or *procedures without connections* are lower level demands, while those involving *procedures with connections* or requiring processes described as *doing mathematics* are high level. Of the 708 examples and exercises, textbooks A, B, C and D contain 46%, 85%, 93%, and 85% of lower cognitive demand tasks respectively. High level demand tasks are associated with active inquiry and the formation of connections that lead to better mathematical learning. However, textbooks A, B, C and D encompassed only a marginal 54%, 15%, 7%, and 15% of high cognitive demand tasks respectively. These tasks were often identified within exercise sections labeled as critical thinking and writing about mathematics. With the exception of textbook A, the hidden message is that teachers and students should be spending a larger portion of their efforts working on lower cognitive demands tasks.

To what extent do texts use and connect mathematical representations? NCTM (2014a) describes five types of representation: visual, physical, symbolic, contextual, and verbal. Flexible and adaptive thinking between representations magnifies mathematical structure and enhances problem solving; therefore, the connections made among the representational forms is equally important.

In our analysis, the primary source of evidence was the examples and solutions given at the beginning of each section and our secondary source was the nature of the representations either requested or given within the end-of-section exercises. Textbook A almost always used multiple representations and frequently made connections between them. Additionally, nearly every example and exercise was contextualized and it frequently connected symbolic and visual representations. In contrast, textbooks B and C heavily emphasized symbolic representation and rarely made explicit connections. Out of fourteen examples in textbook B, only six were in context. It situated contextualized problems toward the end of all exercises. In textbook C, only three of seventeen examples were in context and these were in the last of three sections devoted to linear growth. Textbook D used mostly symbolic representations; of 25 examples, 10 were in context. Textbook D was unique in that it had a single section devoted to graphic or visual representation. In this section, strong explicit connections were made between symbolic, visual and contextual information. Textbook A and D demonstrated that explicit connections are possible but not necessarily commonplace. None of the textbooks mentions using physical models. We found it ironic that a course described as modeling contained so few contextualized problems in many textbooks.

To what extent do the textbooks facilitate meaningful mathematical discourse? Because textbook communication is strictly one-way, mathematical discourse in this arena is dependent on the reader's ability to comprehend what he or she is reading. The *Flesch Reading Ease* statistic quantifies readability by measuring number of syllables per word, sentence length, and other factors (Flesch, 1948). The statistic ranges from 0-100, with higher numbers correlating to easier readability. Similarly, the *Flesch-Kincaid Grade Level* purports to measure the grade level equivalence of selected text (Kincaid, Fishburne, Rogers, & Chissom, 1975).

Each textbook offered an introductory overview of the topic of linear growth, usually in the form of a real world problem that could be modeled by a linear function. We measured the readability of each of these introductory passages and obtained *Flesch Reading Ease* statistics of 40.2, 35.8, 49.7, and 68.3 for textbooks A, B, C and D respectively. The associated *Flesch-Kincaid Grade Level* scores were 12.8, 13.0, 11.7 and 7.9 respectively. Wilkins, Hartman, Howland, and Sharma (2010) found that roughly half of one university's college entrants lacked the literacy skills expected of incoming freshmen. The reading level of most of the textbooks in our sample may exceed the comprehension skills of a significant number of students.

To what extent do the textbooks pose purposeful questioning? We identified the questions contained in the relevant sections of each textbook and classified each as either open or closed. In all four textbooks, questions tended to be closed; that is, a single, correct answer was expected. An exception was that both textbooks C and D employed a *Writing in Mathematics* section in the exercises where open questions offered flexibility in how questions could be correctly answered. Additionally, textbook D included a *Critical Thinking* section in the exercises that also contained some open questions.

Imperatives often stand in for questions in mathematics textbooks. Herbel-Eisenmann (2007) noted it is important to evaluate imperatives (e.g. directives such as to *create a graph* or *consider a pattern*) when evaluating a textbook "because they implicitly address the reader and involve him or her in the construction of mathematics" (p. 349). She classified imperatives as *inclusive* or *exclusive,* as a way to describe how the reader is positioned with respect to the mathematical community. Inclusive imperatives include the reader by directing him or her to, for example, *think, consider,* or *predict*. In contrast, exclusive imperatives metaphorically position the reader as a "scribbler" (Rotman, 1988) by instructing him or her to, for example, *write, make,* or *find*. We found that most imperatives contained in the textbooks were exclusive and that inclusive imperatives were rarely, if ever, used. Interestingly, the exceptions again were the *Writing in Mathematics* and *Critical Thinking* sections of textbooks C and D, which contained more inclusive imperatives than what was found in other parts of the text.

To what extent does the textbook build procedural fluency from conceptual understanding? According to NCTM (2014b), "Procedural fluency builds from an initial exploration and discussion of number concepts to using informal reasoning strategies and the properties of operations to develop general methods for solving problems" ("NCTM Position," para. 1*)*. However, without exception, abstracted mathematics preceded its application in the form of word problems. Varying levels of formality characterized these presentations in the form of precise language and definitions, and the privileging efficient procedures. Additionally, all four textbooks implicitly employed conventional mathematical norms (e.g. the variable belongs on the left hand side of the equation; answers should always be written in simplified form, etc.). Initial explorations and informal reasoning strategies were nonexistent in these textbook formats. As textbooks inform the instructor about what to teach (Fan et al., 2013), building procedural fluency from conceptual understanding is left unaddressed.

To what extent does the textbook support productive struggle in learning mathematics? The phrase "productive struggle" suggests an environment in which the answer is not immediate and in which forethoughts are components of the learning process. We evaluated the extent to which the textbooks supported or suggested such learning environments. We believed that simple, straightforward procedures serve to reduce mathematical struggle, whereas general heuristics offer support without dictating the strategy. With heuristics, students must rely on their own sense making and reasoning skills, which are important components of productive struggle.

Textbooks A and D emphasized heuristics over procedures, although textbook D prescribed procedures at times. Occasionally, textbook D acknowledged alternate pathways to solving equations. In contrast, textbooks B and C predominately prescribed procedures and dictated single solution methods.

To what extent does the textbook elicit and use evidence of student thinking? None of the textbooks in our sample offered insight to how students commonly approach the relevant mathematical ideas or the misconceptions they may hold. However, three of the four textbooks in our study were student editions. It is possible that instructor editions may offer more depth and guidance for instructors, but the one instructor edition we evaluated did not support this supposition. If the instructor is meant to use student thinking to scaffold mathematics learning, these textbooks do not offer such support.

Discussion

As evident in our findings, these textbooks offered few strategic directions for teaching and learning mathematics in ways suggested by NCTM (2014a; 2014b). Indeed, with a few exceptions, the reviewed textbooks provided minimal support for NCTM's elements of effective teaching and learning. All the textbooks portrayed an abstracted, purified form of mathematics which could only be used to solve problems after proficiency was achieved. This approach perpetuates the idea that mathematics is best learned through a progression of algorithmic procedures that gradually increase in cognitive demand and difficulty.

The textbooks implied that students are meant to understand the world *through mathematics* as opposed to the other way around. Thus, students are positioned as dependent on the instructor for access to mathematics, and ultimately, to problem solving. The lack of support for multiple pathways and entry points for tasks, and lack of support of discussion and connected representations, convey the message that these elements are unimportant to student learning.

Mesa and Griffiths (2012) suggested that instructors perceive textbooks as a resource for students. As a potential catalyst for a fresh perspective on mathematics, these textbooks unfortunately transmit traditional views of mathematics to non-STEM students. Fan et al. (2013) noted the role textbooks play in informing instructors what and how to teach. Therefore, instructional reform at the tertiary level should consider textbooks as valuable tools for both instructor and students. Can textbooks mediate curriculum design for instructor and mediate mathematical perceptions for both parties? We suspect the mathematics used in situated contextual learning may be a more productive outlook for the population of students being served and that further research in this area is imperative to affect change.

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